Published by Institute of Physics Publishing for SISSA
Received: *January 17, 2007*
Accepted: *January 29, 2007*
Published: *February 27, 2007*

Unquenched flavors in the Klebanov-Witten model

Francesco Benini and Stefano Cremonesi
SISSA/ISAS and INFN-Sezione di Trieste
Via Beirut 2, I-34014, Trieste, Italy
E-mail: benini@sissa.it, cremones@sissa.it

Felipe Canoura and Alfonso V. Ramallo
Departamento de Fisica de Particulas, Universidade de Santiago de Compostela and
Instituto Galego de Fisica de Altas Enerxias (IGFAE)
E-15782, Santiago de Compostela, Spain
E-mail: canoura@fpaxp1.usc.es, alfonso@fpaxp1.usc.es

Carlos Núñez
Department of Physics, University of Swansea, Singleton Park
Swansea SA2 8PP, United Kingdom
E-mail: c.nunez@swansea.ac.uk

Abstract: Using AdS/CFT, we study the addition of an arbitrary number of backreacting flavors to the Klebanov-Witten theory, making many checks of consistency between our new Type IIB plus branes solution and expectations from field theory. We study generalizations of our method for adding flavors to all $\mathcal{N} = 1$ SCFTs that can be realized on D3-branes at the tip of a Calabi-Yau cone. Also, general guidelines suitable for the addition of massive flavor branes are developed.

Keywords: Brane Dynamics in Gauge Theories, AdS-CFT Correspondence, Gauge-gravity correspondence, D-branes.

Contents

1. Introduction

The AdS/CFT conjecture originally proposed by Maldacena [1] and refined in [2, 3] is one of the most powerful analytic tools for studying strong coupling effects in gauge theories. There are many examples that go beyond the initially conjectured duality and first steps in generalizing it to non-conformal models were taken in [4]. Later, very interesting developments led to the construction of the gauge-string duality in phenomenologically more relevant theories i.e. minimally or non-supersymmetric gauge theories [5].

Conceptually, a clear setup for duals to theories with few SUSY's is obtained by breaking conformality and (perhaps partially) supersymmetry, deforming $\mathcal{N} = 4$ SYM with relevant operators or VEV's. The models put forward in [5] are very good examples of this.

Even when there are important technical differences, in the same line of thought, we can consider the model(s) developed by Klebanov and a distinguished list of physicists: Witten [6], Nekrasov [7], Tseytlin [8], Strassler [9], Herzog and Gubser [10] and Dymarsky and Seiberg [11]. In these papers (and many extensions of them), a far reaching idea has been developed, namely to flow to a confining field theory with minimal SUSY starting from an $\mathcal{N} = 1$ SCFT with a product gauge group $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$, bifundamental chiral matter and a quartic superpotential for the chiral superfields.[1] The superconformal field theory described above rules the low energy dynamics of N_c D3-branes at the tip of the conifold. Then conformality is broken by the addition of fractional branes, that effectively unbalance the ranks of the gauge groups [7, 8]. A "duality cascade" starts and the flow to the IR leaves us with a confining field theory [9]. Subtleties related to the last steps of the cascade have been discussed in [10, 12]. All this interesting physics is very nicely described with great detail in [13].

In this paper we will concentrate on a nonconformal theory without cascade. The starting point is a Type IIB solution dual to an $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$ $\mathcal{N} = 1$ SCFT also known as the Klebanov-Witten field theory/geometry. One of the aims of the paper is to add an arbitrary large number of flavors to each of the gauge groups. The addition of fundamental degrees of freedom is an important step toward the understanding of QCD-like dynamics, in different regions of the space of parameters.

A very fructiferous idea used to add flavors to different field theories (using the string dual) was described in [14] and then applied to various backgrounds, 'flavoring' different dual field theories, in many subsequent publications (for a complete list see citations to [14]). As it was clearly stated in the original paper, the procedure spelled out in [14] consists in the addition of a finite number N_f of spacetime filling flavor D7-branes to the $N_c \to \infty$ color D3-branes extending in the Minkowski directions, and the usual decoupling limit ($g_s \to 0$, $N_c \to \infty$, $g_s N_c$ fixed) of the D3-branes is performed, keeping the number N_f of flavor branes fixed. Then the D3-branes generate the geometry and the flavor branes only minimize their worldvolume Dirac-Born-Infeld action in this background without deforming it. This is the probe limit. In the dual description they are considering the addition of

[1] It is obvious that such a field theory is non-renormalizable and must be thought of as the IR of some UV well defined theory. In [6] a UV completion in terms of an orbifolded $\mathcal{N} = 2$ field theory is given.

a finite number of flavors N_f to the large N_c gauge theory, in the strict double scaling 't Hooft limit ($g_{\rm YM} \to 0$, $N_c \to \infty$, $\lambda = g_{\rm YM}^2 N_c$ fixed). In the lattice literature this is called the 'quenched' approximation: the dynamics of the colors and its effect on the flavors is completely taken into account, but the backreaction of the flavors onto the colors is neglected. In the probe limit this approximation becomes exact.

It is interesting to go beyond this 'quenched' or 'non-backreacting' approximation and see what happens when one adds a large number of flavors, of the same order of the number of colors, and the backreaction effects of the flavor branes are considered. Indeed, many phenomena that cannot be captured by the quenched approximation, might be apparent when a string backreacted background is found.

In this paper we will propose a Type IIB dual to the field theory of Klebanov and Witten, in the case in which a large number of flavors ($N_f \sim N_c$) is added to each gauge group. We will also present interesting generalizations of this to cases describing different duals to $\mathcal{N} = 1$ SCFT's constructed from D3-branes placed at singularities.

Let us briefly describe the procedure we will follow, inspired mostly by the papers [15–17] and more recently [18, 19]. In those papers (dealing with the addition of many fundamentals in the non-critical string and Type IIB string respectively), flavors are added into the dynamics of the dual background via the introduction of N_f spacetime filling flavor branes, whose dynamics is given by a Dirac-Born-Infeld action. This dynamics is intertwined with the usual Einstein-like action of IIB and a new solution is found, up to technical subtleties described below.

1.1 Generalities of the procedure used

To illustrate the way flavor branes will be added, let us start by considering the background of Type IIB supergravity that is conjectured to be dual to the Klebanov-Witten field theory: an $\mathcal{N} = 1$ SCFT with gauge group $\mathrm{SU}(N_c)\times\mathrm{SU}(N_c)$, two chiral multiplets of bifundamental matter $A_i, B_i,\ i = 1, 2$ and a (classically irrelevant) quartic superpotential

$$W = \lambda \mathrm{Tr}(A_i B_j A_k B_l)\, \epsilon^{ik}\epsilon^{jl}\ . \tag{1.1}$$

The dual Type IIB background reads

$$ds^2 = h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2}\Big\{ dr^2 + \frac{r^2}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2) + \tag{1.2}$$
$$+\frac{r^2}{9}(d\psi + \sum_{i=1,2}\cos\theta_i\, d\varphi_i)^2\Big\}$$

$$F_5 = \frac{1}{g_s}(1+*)\, d^4x \wedge dh(r)^{-1}$$
$$h(r) = \frac{27\pi g_s N_c \alpha'^2}{4r^4} \tag{1.3}$$

with constant dilaton and all the other fields in Type IIB supergravity vanishing. The set of coordinates that will be used in the rest of the paper is $x^M =$

$\{x^0, x^1, x^2, x^3, r, \theta_1, \varphi_1, \theta_2, \varphi_2, \psi\}$. For the sake of brevity, in the following we will take units is which $g_s = 1$, $\alpha' = 1$.

We will add N_f spacetime filling D7-branes to this geometry, in a way that preserves some amount of supersymmetry. This problem was studied in [20, 21] for the conformal case and in [22] for the cascading theory. These authors found calibrated embeddings of D7-branes which preserve (at least some fraction of) the supersymmetry of the background. We will choose to put two sets of D7-branes on the surfaces parametrized by

$$\begin{aligned} \xi_1^\alpha &= \{x^0, x^1, x^2, x^3, r, \theta_2, \varphi_2, \psi\} \qquad \theta_1 = \text{const.} \qquad \varphi_1 = \text{const.}\,, \\ \xi_2^\alpha &= \{x^0, x^1, x^2, x^3, r, \theta_1, \varphi_1, \psi\} \qquad \theta_2 = \text{const.} \qquad \varphi_2 = \text{const.}\,. \end{aligned} \tag{1.4}$$

Note that these two configurations are mutually supersymmetric with the background. Moreover, since the two embeddings are noncompact, the gauge theory supported on the D7's has vanishing 4d effective coupling on the Minkowski directions; therefore the gauge symmetry on them is seen as a flavor symmetry by the 4d gauge theory of interest. The two sets of flavor branes introduce a $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ symmetry,[2] the expected flavor symmetry with massless flavors. The configuration with two sets (two branches) can be deformed to a single set, shifted from the origin, that represents massive flavors, and realizes the explicit breaking of the flavor symmetry to the diagonal vector-like $\mathrm{U}(N_f)$. Our configuration (eq. (1.4)) for probes is nothing else than the $z_1 = 0$ holomorphic embedding of [21].

We will then write an action for a system consisting of type IIB supergravity[3] plus D7-branes described by their Dirac-Born-Infeld action (in Einstein frame):

$$\begin{aligned} S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G}\Big[R - \frac{1}{2}\partial_M\phi\partial^M\phi - \frac{1}{2}e^{2\phi}|F_1|^2 - \frac{1}{4}|F_5|^2\Big] + \\ -T_7 \sum^{N_f} \int d^8\xi\, e^\phi \Big[\sqrt{-\hat{G}_8^{(1)}} + \sqrt{-\hat{G}_8^{(2)}}\Big] + T_7 \sum^{N_f} \int \hat{C}_8\,, \end{aligned} \tag{1.5}$$

where we have chosen the normalization $|F_p|^2 = \frac{1}{p!} F_p F_p (G^{-1})^p$.
Notice that we did not excite the worldvolume gauge fields, but this is a freedom of the approach we adopted. Otherwise one may need to find new suitable κ-symmetric embeddings.

These two sets of D7-branes are localized in their two transverse directions, hence the equations of motion derived from (1.5) will be quite complicated to solve, due to the presence of source terms (Dirac delta functions).

But we can take some advantage of the fact that we are adding lots of flavors. Indeed, since we will have many ($N_f \sim N_c$) flavor branes, we might think about distributing them in a homogeneous way on their respective transverse directions. This 'smearing procedure'

[2]The diagonal axial $\mathrm{U}(1)$ is anomalous

[3]The problems with writing an action for Type IIB that includes the self-duality condition are well known. Here, we just mean a Lagrangian from which the equations of motion of Type IIB supergravity are derived. The self-duality condition is imposed on the solutions.

boils down to approximating

$$
\begin{aligned}
T_7 \sum^{N_f} \int d^8\xi \, e^{\phi} \sqrt{-\hat{G}_8^{(i)}} \quad &\to \quad \frac{T_7 N_f}{4\pi} \int d^{10}x \, e^{\phi} \sin\theta_i \sqrt{-\hat{G}_8^{(i)}} \\
T_7 \sum^{N_f} \int \hat{C}_8 \quad &\to \quad \frac{T_7 N_f}{4\pi} \int \Big[Vol(Y_1) + Vol(Y_2) \Big] \wedge C_8 \ ,
\end{aligned}
\tag{1.6}
$$

with $Vol(Y_i) = \sin\theta_i \, d\theta_i \wedge d\varphi_i$ the volume form of the S^2's.

This effectively generates a ten dimensional action

$$
\begin{aligned}
S =& \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \Big[R - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2} e^{2\phi} |F_1|^2 - \frac{1}{4} |F_5|^2 \Big] + \\
& - \frac{T_7 N_f}{4\pi} \int d^{10}x \, e^{\phi} \sum_{i=1,2} \sin\theta_i \sqrt{-\hat{G}_8^{(i)}} + \frac{T_7 N_f}{4\pi} \int \Big[Vol(Y_1) + Vol(Y_2) \Big] \wedge C_8 \ .
\end{aligned}
\tag{1.7}
$$

We can derive in the smeared case the following (not so involved) equations of motion, coming from the action (1.7):

$$
\begin{aligned}
R_{MN} - \frac{1}{2} G_{MN} R =& \frac{1}{2} \Big(\partial_M \phi \partial_N \phi - \frac{1}{2} G_{MN} \partial_P \phi \partial^P \phi \Big) + \frac{1}{2} e^{2\phi} \Big(F_M^{(1)} F_N^{(1)} - \frac{1}{2} G_{MN} |F^{(1)}|^2 \Big) + \\
& + \frac{1}{96} F^{(5)}_{MPQRS} F_N^{(5)PQRS} + T_{MN} \\
D^M \partial_M \phi =& e^{2\phi} |F_1|^2 + \frac{2\kappa_{10}^2 T_7}{\sqrt{-G}} \frac{N_f}{4\pi} e^{\phi} \sum_{i=1,2} \sin\theta_i \sqrt{-\hat{G}_8^{(i)}} \\
d\Big(e^{2\phi} * F_1 \Big) =& 0 \\
dF_1 =& -2\kappa_{10}^2 T_7 \frac{N_f}{4\pi} \Big[Vol(Y_1) + Vol(Y_2) \Big] \\
dF_5 =& 0 \ .
\end{aligned}
\tag{1.8}
$$

The modified Bianchi identity is obtained through $F_1 = -e^{-2\phi} * F_9$, and comes from the WZ part of the action (1.7). The contribution to the stress-energy tensor coming from the two sets of N_f D7 flavor branes is given by

$$
T^{MN} = \frac{2\kappa_{10}^2}{\sqrt{-G}} \frac{\delta S^{\text{flavor}}}{\delta G_{MN}} = -\frac{N_f}{4\pi} \frac{e^{\phi}}{\sqrt{-G}} \sum_{i=1,2} \sin\theta_i \frac{1}{2} \sqrt{-\hat{G}_8^{(i)}} \hat{G}_8^{(i)\alpha\beta} \delta_\alpha^M \delta_\beta^N \ ,
\tag{1.9}
$$

where α, β are coordinate indices on the D7. In the subsequent sections we will solve the equations of motion (1.8)–(1.9) and will propose that this Type IIB background is dual to the Klebanov-Witten field theory when two sets of N_f flavors are added for each gauge group. We will actually find BPS equations for the purely bosonic background, by imposing that the variations of the dilatino and gravitino vanish. We will verify that these BPS first-order equations solve all the equations of motion (1.8).

Let us add some remarks on some important points about the resolution of the system. First of all, it is clear from the Bianchi identity of F_1 in (1.8) that we will not be able to define the axion field C_0 on open subsets.

Regarding the solution of the equations of motion, we will proceed by proposing a *deformed background* ansatz of the form

$$
\begin{aligned}
ds^2 &= h^{-1/2} dx_{1,3}^2 + h^{1/2} \Big\{ dr^2 + \frac{e^{2g}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2) + \frac{e^{2f}}{9} (d\psi + \sum_{i=1,2} \cos\theta_i \, d\varphi_i)^2 \Big\} \\
F_5 &= (1 + *) \, d^4x \wedge K \, dr \qquad (1.10) \\
F_1 &= \frac{N_f}{4\pi} (d\psi + \cos\theta_2 \, d\varphi_2 + \cos\theta_1 \, d\varphi_1) \, .
\end{aligned}
$$

Thanks to the smearing procedure, all the unknown function h, f, g, K and the dilaton ϕ only depend on the radial coordinate r.

The Bianchi identity for the five-form field-strength gives

$$
K \, h^2 \, e^{4g+f} = 27\pi N_c \, , \qquad (1.11)
$$

and we will obtain solutions to (1.8) by imposing that the BPS equations derived from the vanishing of the gravitino and gaugino variations and the Bianchi identities are satisfied. These will produce ordinary first-order equations for $f(r)$, $g(r)$, $h(r)$, $K(r)$, $\phi(r)$. We will also be able to derive these BPS equations from a superpotential in the reduction of Type IIB supergravity.

We will study in detail the dual field theory to the supergravity solutions mentioned above, making a considerable number of matchings. The field theories turn out to have positive β-function along the flow, exhibiting a Landau pole in the UV. In the IR we still have a strongly coupled field theory, which is "almost conformal". We will also generalize all these results to the interesting case of a large class of different $\mathcal{N} = 1$ SCFTs, deformed by the addition of flavors. In particular we will be able to add flavors to every gauge theory whose dual is $AdS_5 \times M_5$, where M_5 is a five-dimensional Sasaki-Einstein manifold. New solutions will be found that describe the 'unflavored' case, making contact with old results. Finally, a possible way of handling the massive flavor case is undertaken.

We have explained the strategy we adopt to add flavors, so this is perhaps a good place to discuss some interesting issues. The reader might be wondering about the 'smearing procedure' discussed above, what is its significance and effect on the dual gauge theory, among other questions. It is clear that we smear the flavor branes just to be able to write a 10-dimensional action that will produce ordinary (in contrast to partial) differential equations without Dirac delta functions source terms.

The results we will show and the experience obtained in [17, 18] show that many properties of the flavored field theory are still well captured by the solutions obtained following the procedure described above. It is not clear what important phenomena on the gauge theory we are losing in smearing, but see below for an important subtlety.

One relevant point to discuss is related to global symmetries. Let us go back to the weak coupling ($g_s N_c \to 0$) limit, in which we have branes living on a spacetime that is the product of four Minkowski directions and the conifold. When all the flavor branes of the two separate stacks (1.4) are on top of each other, the gauge symmetry on the D7's worldvolume is given by the product $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$. When we take the decoupling

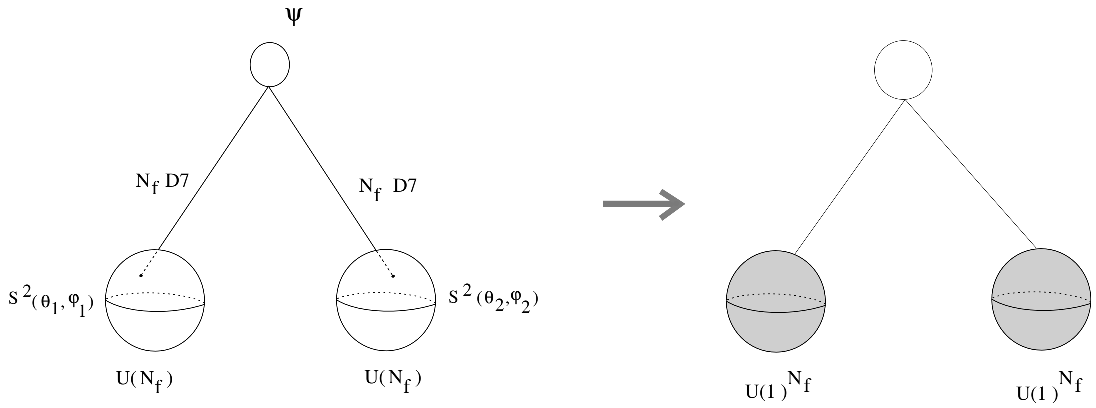


Figure 1: We see on the left side the two stacks of N_f flavor-branes localized on each of their respective S^2's (they wrap the other S^2). The flavor group is clearly $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$. After the smearing on the right side of the figure, this global symmetry is broken to $\mathrm{U}(1)^{N_f-1} \times \mathrm{U}(1)^{N_f-1} \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A$.

limit for the D3-branes $\alpha' \to 0$, with fixed $g_s N_c$ and keeping constant the energies of the excitations on the branes, we are left with a solution of Type IIB supergravity that we propose is dual to the Klebanov-Witten field theory with N_f flavors for both gauge groups [21]. In this case the flavor symmetry is $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$, where the axial $\mathrm{U}(1)$ is anomalous. This background would be for sure very involved, since it would depend on the coordinates (r, θ_1, θ_2), if the embeddings of the two stacks of D7-branes are $\theta_1 = 0$ and $\theta_2 = 0$, respectively. When we smear the N_f D7-branes, we are breaking $\mathrm{U}(N_f) \to \mathrm{U}(1)^{N_f}$ (see figure 1).

There is one important point to contrast with [17]. In that paper, a smearing is also proposed but it is argued that the dual field theory (a version of $\mathcal{N} = 1$ SQCD with a quartic superpotential in the quark superfields) possesses $\mathrm{U}(N_f)$ global flavor symmetry. As in all backgrounds constructed on wrapped branes, the effects of the Kaluza-Klein modes play an important rôle and the dual field theory behaves as 4-dimensional only in the far IR.[4] In this regime, when the internal manifold shrinks to small size, for energies below this inverse size we do not effectively see the breaking $\mathrm{U}(N_f) \to \mathrm{U}(1)^{N_f}$. In contrast, the backgrounds obtained by placing D-branes at conical singularities, like [6]–[10] as well as our solution, describe a four dimensional field theory all along the flow.

It might be interesting for the reader to note that the papers in the line of [14] are working in the context of 't Hooft expansion [24]. When the ratio N_f/N_c is very small, one can ignore the backreaction effects of the flavor branes on the geometry. This is the dual version to the suppression of effects that include the running of fundamentals in internal loops. Even when these fundamentals are massless, their effects while running in loops are suppressed by a factor of $\mathcal{O}(N_f/N_c)$. But in the strict 't Hooft limit, if the number of flavors is kept fixed, the corrections due to the quantum dynamics of quarks exactly vanish.

[4] For a detailed study of the role and dynamics of the KK modes in wrapped brane setups, see [23].

In the cases considered in [17, 18], the ratio above is of order one and we are working on the so called Veneziano's topological expansion [25]. New physics (beyond the 't Hooft limit) is captured by Veneziano's proposal: we will be able to see this in the present paper that considers the backreaction of the flavor branes, just like was observed in [17, 18], in contrast with the papers that worked in the 't Hooft approximation as proposed in [14].

Another point that is worth elaborating on is whether there is a limit on the number of D7-branes that can be added. Indeed, since a D7-brane is a codimension-two object (like a vortex in $2+1$ dimensions) its gravity solution will generate a deficit angle; having many seven branes, will basically "eat-up" the transverse space. This led to the conclusion that solutions that can be globally extended cannot have more than a maximum number of twelve D7-branes [26] (and exactly twenty-four in compact spaces). In this paper we are adding a number $N_f \to \infty$ of D7-branes, certainly larger that the bound mentioned above. Like in the papers [27, 28], we will adopt the attitude of analyzing the behavior of our solutions and we will see that they give sensible results. But there is more than that: the smearing procedure distributes the D7's all over this 2-dimensional compact space, in such a way that the equation for the axion-dilaton is not the one in the vacuum at any point. This avoids the constraint on the number of D7-branes, which came from solving the equation of motion for the axion-dilaton outside sources.

Finally, we must emphasize that this is not the first paper that deals with the D3/D7 system in the context of "AdS/CFT with flavors". Indeed, very good papers have been written where this problem was faced looking for a solution where the flavor branes are replaced by fluxes in terms of the Type IIB supergravity fields, dilaton and an axion (ϕ, C_0). The BPS equations for the D3/D7 system in cases preserving 8 supercharges were written in [27, 28], a partially explicit solution of the equations of motion in the presence of sources was found in [29] for the orbifold case, more interesting geometrical aspects were discussed in [30] and an involved solution was found in [31], where some matching with gauge theory behavior was attempted.[5]

The papers [27 − 31] were written with the idea of letting the flavor branes backreact. One qualitative difference with respect to what we explained above is that the authors of [27, 28, 30, 31] consider the case in which D3-branes are added in the background produced by D7-branes and solve the Laplace equation, in this case for the deformation introduced by the D3's. In contrast, we consider the background produced by the D3-branes and we deform it to take into account the "smeared" backreaction of the D7-branes. The two procedures are different.

One advantage of the approach proposed in [17] is that the flavor degrees of freedom explicitly appear in the DBI action that allows the introduction of SU(N_f) gauge fields in the bulk that are dual to the global symmetry in the dual field theory, while it is difficult to see how they will appear in a Type IIB solution that only includes RR fluxes. Our approach produces a simple SUSY solution to (1.8) and the analysis of gauge theory effects is simple to do. Besides, the proposal of [17] used in the present work is the natural continuation

[5]Even though slightly unrelated to the D3/D7 system, we cannot resist here to mention the beautiful solution found by Cherkis and Hashimoto for a localized D2/D6 system [32].

of the many successful results obtained in papers in the line of [14]. Indeed, we are just following the idea of [14] for a large number of flavor branes.

1.2 Organization of this paper

This paper is organized in two main parts. In part I we will present the addition of flavors to the Klebanov-Witten solution. A detailed analysis of the supergravity plus branes solutions and the study of the dual gauge theory, as read from the above mentioned solutions, is performed. A reader mainly interested with the line of research, but who does not want to go in full details, should be happy reading this introduction, part I and appendix C.

The readers who intend to work on this subject and want to study these results in more technical detail or want to appreciate the beauty and generality in our formalism are referred to part II. Also in part II the reader will find a sketch of how to deal with massive flavors using these techniques.

Those readers who are not attracted by the physics of flavor using AdS/CFT techniques, but just want to learn about some new solutions (born out of our 'deformed backgrounds' as described above), should read the introduction and the appendix B.

Some other appendices complement nicely our presentation.

The section of conclusions includes also a summary of results and proposes future directions that the interested reader might want to pursue.

2. Part I: adding flavors to the Klebanov-Witten field theory

2.1 What to expect from field theory considerations

In this first part we will address in detail the problem of adding a large number of back-reacting non-compact D7-branes to the Klebanov-Witten Type IIB supergravity solution, which describes D3-branes at the tip of the conifold. Before presenting the solution and describing how it is obtained, we would like to have a look at the dual field theory, and sketch which are the features we expect.

For this purpose, we consider the case of probe D7-branes, and mainly summarize what was pointed out in [21]. The conifold is a non-compact Calabi-Yau 3-fold, defined by one equation in $\mathbb{C}^4$:

$$z_1 z_2 - z_3 z_4 = 0 \,. \tag{2.1}$$

Since this equation is invariant under a real rescaling of the variables, the conifold is a real cone, whose base is the Sasaki-Einstein space $T^{1,1}$ [6, 33]. It can be shown that $T^{1,1}$ is a U(1) bundle over the Kähler-Einstein space $S^2 \times S^2$, and that its isometry group is $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

Klebanov and Witten [6] obtained an interesting example of gauge/gravity duality by placing a stack of N_c D3-branes at the apex of the conifold. The branes source the RR 5-form flux and warp the geometry, giving the Type IIB supergravity solution (1.2). The dual field theory, describing the IR dynamics on the worldvolume of the branes, has gauge group $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$ and matter fields A_i, B_i, $i = 1, 2$ which transform in the bifundamental representations $(N_c, \overline{N_c})$ and $(\overline{N_c}, N_c)$ respectively. The theory has also a quartic superpotential $W_{\rm KW} = \lambda\,{\rm Tr}(A_i B_j A_k B_l)\,\epsilon^{ik}\epsilon^{jl}$. The field theory is $\mathcal{N}=1$ superconformal, and the anomaly-free U(1) R-symmetry of the superconformal algebra is dual to the U(1) isometry of the fiber in $T^{1,1}$, generated by the so-called Reeb vector. In the algebraic definition (2.1) it is realized as a common phase rotation of the four coordinates: $z_i \to e^{-i\alpha} z_i$.

The addition of flavors, transforming in the fundamental and antifundamental representations of the gauge groups, can be addressed by including probe D7-branes into the geometry, following the procedure proposed in [14]. This was done in [21], where the embedding of the flavor branes and the corresponding superpotential for the fundamental and antifundamental superfields were found. The D7-branes have four Minkowski directions parallel to the stack of D3-branes transverse to the conifold, whereas the other four directions are embedded holomorphically in the conifold. In particular, D7-branes describing massless flavors can be introduced by considering the holomorphic noncompact embedding $z_1 = 0$. The flavors, which correspond to 3-7 and 7-3 strings, are massless because the D7-branes intersect the D3-branes. Note that the D7-branes have two branches, described by $z_1 = z_3 = 0$ and $z_1 = z_4 = 0$, each one corresponding to a stack. The presence of two branches is required by RR tadpole cancellation: in the field theory this amounts to adding flavors in vector-like representations to each gauge group, hence preventing gauge anomalies. The fundamental and antifundamental chiral superfields of the two gauge groups will be denoted as q, $\tilde{q}$ and Q, $\tilde{Q}$ respectively, and the gauge invariant and flavor invariant superpotential proposed in [21] is

$$W = W_{\rm KW} + W_f\ , \tag{2.2}$$

where

$$W_{\rm KW} = \lambda\,{\rm Tr}(A_i B_k A_j B_l)\,\epsilon^{ij}\epsilon^{kl} \tag{2.3}$$

is the SU(2)$\times$SU(2) invariant Klebanov-Witten superpotential for the bifundamental fields. For a stack of flavor branes, it is conventional to take the coupling between bifundamentals and quarks at a given point of S^2 as

$$W_f = h_1\,\tilde{q}^a A_1 Q_a + h_2\,\tilde{Q}^a B_1 q_a\,. \tag{2.4}$$

This coupling between bifundamental fields and the fundamental and antifundamental flavors arises from the D7 embedding $z_1 = 0$. The explicit indices are flavor indices. This superpotential, as well as the holomorphic embedding $z_1 = 0$, explicitly breaks the SU(2)$\times$SU(2) global symmetry (this global symmetry will be recovered after the smearing).

The field content and the relevant gauge and flavor symmetries of the theory are summarized in table 1 and depicted in the quiver diagram in figure 2.

The $\mathrm{U}(1)_R$ R-symmetry is preserved at the classical level by the inclusion of D7-branes embedded in such a way to describe massless flavors, as can be seen from the fact that the equation $z_1 = 0$ is invariant under the rotation $z_i \to e^{-i\alpha} z_i$ and the D7 wrap the R-symmetry circle. Nevertheless the $\mathrm{U}(1)_R$ turns out to be anomalous after the addition of flavors, due to the nontrivial C_0 gauge potential sourced by the D7. The baryonic symmetry $\mathrm{U}(1)_B$ inside the flavor group is anomaly free, being vector-like.

JHEP02(2007)090

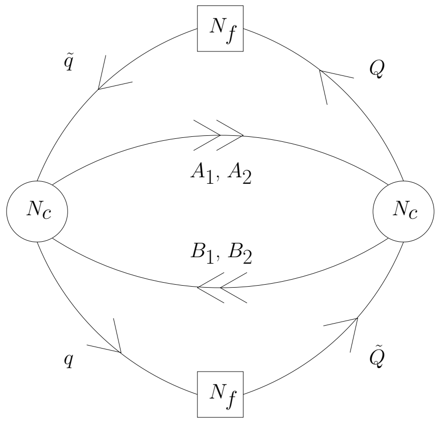


Figure 2: Quiver diagram of the Klebanov-Witten gauge theory with flavors. Circles are gauge groups while squares are non-dynamical flavor groups.

	$\mathrm{SU}(N_c)^2$	$\mathrm{SU}(N_f)^2$	$\mathrm{SU}(2)^2$	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_{B'}$
A	$(N_c, \overline{N_c})$	$(1,1)$	$(2,1)$	$1/2$	0	1
B	$(\overline{N_c}, N_c)$	$(1,1)$	$(1,2)$	$1/2$	0	-1
q	$(N_c, 1)$	$(\overline{N_f}, 1)$	$(1,1)$	$3/4$	1	1
$\tilde{q}$	$(\overline{N_c}, 1)$	$(1, N_f)$	$(1,1)$	$3/4$	-1	-1
Q	$(1, N_c)$	$(1, \overline{N_f})$	$(1,1)$	$3/4$	1	0
$\tilde{Q}$	$(1, \overline{N_c})$	$(N_f, 1)$	$(1,1)$	$3/4$	-1	0

Table 1: Field content and symmetries of the KW field theory with massless flavors.

As was noted in [21], the theory including D7-brane probes is also invariant under a rescaling $z_i \to \beta z_i$, therefore the field theory is scale invariant in the probe approximation. In this limit the scaling dimension of the bifundamental fields is 3/4 and the one of the flavor fields is 9/8, as required by power counting in the superpotential. Then the beta function for the holomorphic gauge couplings in the Wilsonian scheme is

$$\beta_{\frac{8\pi^2}{g_i^2}} = -\frac{16\pi^2}{g_i^3}\beta_{g_i} = -\frac{3}{4}N_f \qquad\qquad \beta_{\lambda_i} = \frac{1}{(4\pi)^2}\frac{3N_f}{2N_c}\lambda_i^2\,, \tag{2.5}$$

with $\lambda_i = g_i^2 N_c$ the 't Hooft couplings. In the strict planar 't Hooft limit (zero order in N_f/N_c), the field theory has a fixed point specified by the afore-mentioned choice of scaling dimensions, because the beta functions of the superpotential couplings and the 't Hooft couplings are zero. As soon as N_f/N_c corrections are taken into account, the field theory has no fixed points for nontrivial values of all couplings. Rather it displays a "near conformal point" with vanishing beta functions for the superpotential couplings, but non-vanishing beta functions for the 't Hooft couplings. In a N_f/N_c expansion, formula (2.5) holds at order N_f/N_c *if* the anomalous dimensions of the bifundamental fields A_j and B_j do not get corrections at this order. *A priori* it is difficult to expect such a behavior from string theory, since the energy-momentum tensor of the flavor branes will induce backreaction effects on the geometry at linear order in N_f/N_c, differently from the fluxes, which will backreact at order $(N_f/N_c)^2$.

Moreover, since we are adding flavors to a conformal theory, we can naively expect a Landau pole to appear in the UV. Conversely, we expect the theory to be slightly away from conformality in the far IR.

2.2 The setup and the BPS equations

The starting point for adding backreacting branes to a given background is the identification of the supersymmetric embeddings in that background, that is the analysis of probe branes. In [20], by imposing κ-symmetry on the brane world-volume, the following supersymmetric embeddings for D7-branes on the Klebanov-Witten background were found:

$$\begin{aligned} \xi_1^\alpha &= \{x^0, x^1, x^2, x^3, r, \theta_2, \varphi_2, \psi\} \qquad \theta_1 = \text{const.} \qquad \varphi_1 = \text{const.} \\ \xi_2^\alpha &= \{x^0, x^1, x^2, x^3, r, \theta_1, \varphi_1, \psi\} \qquad \theta_2 = \text{const.} \qquad \varphi_2 = \text{const.} \end{aligned} \tag{2.6}$$

They are precisely the two branches of the supersymmetric embedding $z_1 = 0$ first proposed in [21]. Each branch realizes a $\mathrm{U}(N_f)$ symmetry group, giving the total flavor symmetry group $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ of massless flavors (a diagonal axial $\mathrm{U}(1)_A$ is anomalous in field theory, which is dual to the corresponding gauge field getting massive in string theory through Green-Schwarz mechanism). We choose these embeddings because of the following properties: they reach the tip of the cone and intersect the color D3-branes; wrap the $\mathrm{U}(1)_R$ circle corresponding to rotations $\psi \to \psi + \alpha$; are invariant under radial rescalings. So they realize in field theory massless flavors, without breaking explicitly the $\mathrm{U}(1)_R$ and the conformal symmetry. Actually, they are both broken by quantum effects. Moreover the configuration does not break the $\mathbb{Z}_2$ symmetry of the conifold solution which corresponds to exchanging the two gauge groups.

The fact that we must include both the branches is due to D7-charge tadpole cancellation, which is dual to the absence of gauge anomalies in field theory. An example of a (non-singular) 2-submanifold in the conifold geometry is $\mathcal{D}_2 = \{\theta_1 = \theta_2,\ \varphi_1 = 2\pi - \varphi_2,\ \psi = \text{const},\ r = \text{const}\}$. The charge distributions of the two branches are

$$\omega^{(1)} = \sum\nolimits_{N_f} \delta^{(2)}(\theta_1, \varphi_1)\, d\theta_1 \wedge d\varphi_1 \qquad\qquad \omega^{(2)} = \sum\nolimits_{N_f} \delta^{(2)}(\theta_2, \varphi_2)\, d\theta_2 \wedge d\varphi_2 \ , \tag{2.7}$$

where the sum is over the various D7-branes, possibly localized at different points, and a correctly normalized scalar delta function (localized on an 8-submanifold) is $\delta^{(2)}(x)\sqrt{-\hat{G}_8}/\sqrt{-G}$. Integrating the two D7-charges on the 2-submanifold we get:

$$\int_{\mathcal{D}_2} \omega^{(1)} = -N_f \qquad\qquad \int_{\mathcal{D}_2} \omega^{(2)} = N_f \ . \tag{2.8}$$

Thus, whilst the two branches have separately non-vanishing tadpole, putting an equal number of them on the two sides the total D7-charge cancels. This remains valid for all (non-singular) 2-submanifolds.

The embedding can be deformed into a single D7 that only reaches a minimum radius, and realizes a merging of the two branches. This corresponds to giving mass to flavors and explicitly breaking the flavor symmetry to $\mathrm{SU}(N_f)$ and the R-symmetry completely. These embeddings were also found in [20].

Each embedding preserves the same four supercharges, irrespectively to where the branes are located on the two 2-spheres parameterized by (θ_1, φ_1) and (θ_2, φ_2). Thus we can smear the distribution and still preserve the same amount of supersymmetry. The 2-form charge distribution is readily obtained to be the same as the volume forms on the two 2-spheres in the geometry, and through the modified Bianchi identity it sources the flux F_1.[6] We expect to obtain a solution where all the functions have only radial dependence. Moreover we were careful in never breaking the $\mathbb{Z}_2$ symmetry that exchanges the two spheres. The natural ansatz is:

$$\begin{aligned} ds^2 = h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2} \Big\{ dr^2 + \\ + \frac{e^{2g(r)}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2) + \frac{e^{2f(r)}}{9} (d\psi + \sum_{i=1,2} \cos\theta_i \, d\varphi_i)^2 \Big\} \end{aligned} \tag{2.9}$$

$$\phi = \phi(r) \tag{2.10}$$

$$F_5 = K(r)\, h(r)^{3/4} \Big(e^{x^0 x^1 x^2 x^3 r} - e^{\theta_1 \varphi_1 \theta_2 \varphi_2 \psi} \Big) \tag{2.11}$$

$$F_1 = \frac{N_f}{4\pi} \big(d\psi + \cos\theta_1 \, d\varphi_1 + \cos\theta_2 \, d\varphi_2 \big) = \frac{3N_f}{4\pi} \, h(r)^{-1/4} e^{-f(r)} \, e^{\psi} \tag{2.12}$$

$$dF_1 = -\frac{N_f}{4\pi} \big(\sin\theta_1 \, d\theta_1 \wedge d\varphi_1 + \sin\theta_2 \, d\theta_2 \wedge d\varphi_2 \big) \,, \tag{2.13}$$

where the unknown functions are $h(r)$, $g(r)$, $f(r)$, $\phi(r)$ and $K(r)$. The angular coordinates θ_i are defined in $[0, \pi]$ while the others have fundamental domain $\varphi_i \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$ with appropriate patching rules.[7] The vielbein is:

$$\begin{aligned} e^{x^i} &= h^{-1/4} \, dx^i & e^r &= h^{1/4} \, dr \\ e^{\theta_i} &= \frac{1}{\sqrt{6}} h^{1/4} e^g \, d\theta_i & e^{\varphi_i} &= \frac{1}{\sqrt{6}} h^{1/4} e^g \, \sin\theta_i d\varphi_i \\ e^{\psi} &= \frac{1}{3} h^{1/4} e^f \, (d\psi + \cos\theta_1 \, d\varphi_1 + \cos\theta_2 \, d\varphi_2) \,. \end{aligned} \tag{2.14}$$

[6] The modified Bianchi identity of F_1 is obtained from the Wess-Zumino action term with $F_1 = -e^{-2\phi} * F_9$.

[7] The correct patching rules on $T^{1,1}$ in the coordinates of (1.2) are:

$$\psi \equiv \psi + 4\pi \,, \qquad \begin{pmatrix} \varphi_1 \\ \psi \end{pmatrix} \equiv \begin{pmatrix} \varphi_1 + 2\pi \\ \psi + 2\pi \end{pmatrix} \,, \qquad \begin{pmatrix} \varphi_2 \\ \psi \end{pmatrix} \equiv \begin{pmatrix} \varphi_2 + 2\pi \\ \psi + 2\pi \end{pmatrix} \,.$$

In fact the space is a $\mathrm{U}(1)$ fibration over $S^2 \times S^2$. The first identification is just the one of the fiber. On the base 2-spheres we must identify the angular variables according to $\varphi_i \equiv \varphi_i + 2\pi$, but this could be accompanied by a shift in the fiber. To understand it, draw the very short (in proper length) path around the point $\theta_1 = 0$: $\theta_1 \ll 1$, $\varphi_1 = t = 4\pi - \psi$ with $t \in [0, 2\pi]$ a parameter along the path. To make it closed, a rotation in φ_1 must be accompanied by an half-rotation in ψ. This gives the second identification.

With this ansatz the field equation $d\big(e^{2\phi} * F_1\big) = 0$ is automatically satisfied, as well as the self-duality condition $F_5 = *F_5$. The Bianchi identity $dF_5 = 0$ gives:

$$K\, h^2\, e^{4g+f} = 27\pi N_c \,, \tag{2.15}$$

and $K(r)$ can be solved. The previous normalization comes from Dirac quantization of the D3-brane charge:

$$\int_{T^{1,1}} F_5 = 2\kappa_{10}^2 T_3\, N_c = (2\pi)^4 N_c \,, \tag{2.16}$$

using a suitable orientation for the volume form of the $T^{1,1}$ space and the fact that $Vol(T^{1,1}) = \frac{16}{27}\pi^3$.

We impose that the ansatz preserves the same four supersymmetries as the probe D7-branes on the Klebanov-Witten solution. With this purpose, let us write the supersymmetric variations of the dilatino and gravitino in type IIB supergravity. For a background of the type we are analyzing, these variations are:

$$\begin{aligned}
\delta_\epsilon\, \lambda &= \frac{1}{2}\, \Gamma^M \left(\partial_M\, \phi \,-\, ie^{\phi}\, F^{(1)}_M \right) \epsilon \\
\delta_\epsilon\, \psi_M &= \nabla_M\, \epsilon \,+\, i\, \frac{e^{\phi}}{4} F^{(1)}_M\, \epsilon \,+\, \frac{i}{1920} F^{(5)}_{PQRST}\, \Gamma^{PQRST}\, \Gamma_M\, \epsilon \,,
\end{aligned} \tag{2.17}$$

where we have adopted the formalism in which ϵ is a complex Weyl spinor of fixed ten-dimensional chirality (see appendix A). It turns out (see section 3.2) that the Killing spinors ϵ (which solve the equations $\delta_\epsilon\, \lambda = \delta_\epsilon\, \psi_M = 0$) in the frame basis (2.14) can be written as:

$$\epsilon = h^{-\frac{1}{8}}\, e^{-\frac{i}{2}\psi}\, \eta \tag{2.18}$$

where η is a constant spinor which satisfies

$$\begin{aligned}
&\Gamma_{x^0x^1x^2x^3}\, \eta = -i\eta \\
&\Gamma_{\theta^1\varphi^1}\, \eta = \Gamma_{\theta^2\varphi^2}\, \eta = i\eta \,, \qquad\qquad \Gamma_{r\psi}\, \eta = -i\eta \,.
\end{aligned} \tag{2.19}$$

Moreover, from (2.17) we get the following system of first-order BPS differential equations:

$$\begin{cases}
g' = e^{f-2g} \\
f' = e^{-f}(3 - 2e^{2f-2g}) - \dfrac{3N_f}{8\pi} e^{\phi-f} \\
\phi' = \dfrac{3N_f}{4\pi} e^{\phi-f} \\
h' = -27\pi N_c\, e^{-f-4g}
\end{cases} \tag{2.20}$$

Notice that taking $N_f = 0$ in the BPS system (2.20) we simply get equations for a deformation of the Klebanov-Witten solution without any addition of flavor branes. Solving the system we find both the original KW background and the solution for D3-branes at a conifold singularity, as well as other solutions which correspond on the gauge theory side to giving VEV to dimension 6 operators. These solutions were considered in [34, 35], and are shown to follow from our system in appendix B.

In order to be sure that the BPS equations (2.20) capture the correct dynamics, we have to check that the Einstein, Maxwell and dilaton equations are solved. This can be done even before finding actual solutions of the BPS system. We checked that the first-order system (2.20) (and the Bianchi identity) in fact *implies* the second order Einstein, Maxwell and dilaton differential equations. An analytic general proof will be given in section 3.3. In the coordinate basis the stress-energy tensor (1.9) is computed to be:

$$\begin{aligned} T_{\mu\nu} &= -\frac{3N_f}{2\pi} h^{-1} e^{\phi-2g}\,\eta_{\mu\nu} && (2.21)\\ T_{rr} &= -\frac{3N_f}{2\pi} e^{\phi-2g} && (2.22)\\ T_{\theta_i\theta_i} &= -\frac{N_f}{8\pi} e^{\phi} && (2.23)\\ T_{\varphi_i\varphi_i} &= -\frac{N_f}{24\pi} e^{\phi-2g}\Big[4e^{2f}\cos^2\theta_i + 3e^{2g}\sin^2\theta_i\Big] && (2.24)\\ T_{\varphi_1\varphi_2} &= -\frac{N_f}{6\pi} e^{\phi+2f-2g}\cos\theta_1\cos\theta_2 && (2.25)\\ T_{\varphi_i\psi} &= -\frac{N_f}{6\pi} e^{\phi+2f-2g}\cos\theta_i && (2.26)\\ T_{\psi\psi} &= -\frac{N_f}{6\pi} e^{\phi+2f-2g}\,. && (2.27) \end{aligned}$$

It is correctly linear in N_f. We did not explicitly check the Dirac-Born-Infeld equations for the D7-brane distribution. We expect them to be solved because of κ-symmetry (supersymmetry) on their world-volume.

Solution with general couplings. We can generalize our set of solutions by switching on non-vanishing VEVs for the bulk gauge potentials C_2 and B_2. We show that this can be done without modifying the previous set of equations, and the two parameters are present for every solution of them. The condition is that the gauge potentials are flat, that is with vanishing field-strength. They correspond thus to (higher rank) Wilson lines for the corresponding bundles.

Let us switch on the following fields:

$$C_2 = c\,\omega_2 \qquad\qquad B_2 = b\,\omega_2\,, \tag{2.28}$$

where the 2-form ω_2 is Poincaré dual to the 2-cycle $\mathcal{D}_2$:

$$\mathcal{D}_2 = \{\theta_1=\theta_2,\ \varphi_1 = 2\pi-\varphi_2,\ \psi=\text{const},\ r=\text{const}\} \tag{2.29}$$

$$\omega_2 = \frac{1}{8\pi}\big(\sin\theta_1\, d\theta_1\wedge d\varphi_1 - \sin\theta_2\, d\theta_2\wedge d\varphi_2\big)\,, \qquad \int_{\mathcal{D}_2}\omega_2 = 1\,. \tag{2.30}$$

We see that $F_{(3)}=0$ and $H_{(3)}=0$. So the supersymmetry variations are not modified, neither are the gauge invariant field-strength definitions. In particular the BPS system (2.20) does not change.

Consider the effects on the action (the argument is valid both for localized and smeared branes). It can be written as a bulk term plus the D7-brane terms:

$$S = S_{bulk} - T_7\int d^8\xi\, e^{\phi}\sqrt{-\det(\hat{G}_8+\mathcal{F})} + T_7\int\Big[\sum\nolimits_q \hat{C}_q\wedge e^{\mathcal{F}}\Big]_8\,, \tag{2.31}$$

with $\mathcal{F} = \hat{B}_2 + 2\pi\alpha' F$ is the D7 gauge invariant field-strength, and hat means pulled-back quantities. To get solutions of the κ-symmetry conditions and of the equations of motion, we must take F such that

$$\mathcal{F} = \hat{B}_2 + 2\pi\alpha' F = 0\,. \tag{2.32}$$

Notice that there is a solution for F because B_2 is flat: $d\hat{B}_2 = \widehat{dB_2} = 0$. With this choice κ-symmetry is preserved as before, since it depends on the combination $\mathcal{F}$. The dilaton equation is fulfilled. The Bianchi identities and the bulk field-strength equations of motion are not modified, since the WZ term only sources C_8. The energy momentum tensor is not modified, so the Einstein equations are fulfilled. The last steps are the equations of B_2 and A_1 (the gauge potential on the D7). For this notice that they can be written:

$$d\frac{\delta S}{\delta F} = 2\pi\alpha'\, d\frac{\delta S_{\text{brane}}}{\delta \mathcal{F}} = 0 \tag{2.33}$$

$$\frac{\delta S}{\delta B_2} = \frac{\delta S_{\text{bulk}}}{\delta B_2} + \frac{\delta S_{\text{brane}}}{\delta \mathcal{F}} = 0\,. \tag{2.34}$$

The first is solved by $\mathcal{F} = 0$ since in the equation all the terms are linear or higher order in $\mathcal{F}$. This is because the brane action does not contain terms linear in $\mathcal{F}$, and this is true provided $C_6 = 0$ (which in turn is possible only if C_2 is flat). The second equation then reduces to $\frac{\delta S_{\text{bulk}}}{\delta B_2} = 0$, which amounts to $d(e^{-\phi} * H_3) = 0$ and is solved.

As we will see in section 2.5, being able to switch on arbitrary constant values c and b for the (flat) gauge potentials, we can freely tune the two gauge couplings (actually the two renormalization invariant scales Λ's) and the two theta angles [6, 36]. This turns out to break the $\mathbb{Z}_2$ symmetry that exchanges the two gauge groups, even if the breaking is mild and only affects C_2 and B_2, while the metric and all the field-strength continue to have that symmetry. However this does not modify the behavior of the gauge theory.

2.3 The solution in type IIB supergravity

The BPS system (2.20) can be solved through the change of radial variable

$$e^f \frac{d}{dr} \equiv \frac{d}{d\rho} \qquad \Rightarrow \qquad e^{-f} dr = d\rho\,. \tag{2.35}$$

We get the new system:

$$\dot{g} = e^{2f-2g} \tag{2.36}$$

$$\dot{f} = 3 - 2e^{2f-2g} - \frac{3N_f}{8\pi} e^{\phi} \tag{2.37}$$

$$\dot{\phi} = \frac{3N_f}{4\pi} e^{\phi} \tag{2.38}$$

$$\dot{h} = -27\pi N_c\, e^{-4g}\,, \tag{2.39}$$

where derivatives are taken with respect to ρ.

Equation (2.38) can be solved first. By absorbing an integration constant in a shift of the radial coordinate ρ, we get

$$e^{\phi} = -\frac{4\pi}{3N_f}\frac{1}{\rho} \qquad \Rightarrow \qquad \rho < 0\,. \tag{2.40}$$

The solution is thus defined only up to a maximal radius $\rho_{\mathrm{MAX}} = 0$ where the dilaton diverges. As we will see, it corresponds to a Landau pole in the ultraviolet (UV) of the gauge theory. On the contrary for $\rho \to -\infty$, which corresponds in the gauge theory to the infrared (IR), the string coupling goes to zero. Note however that the solution could stop at a finite negative ρ_{MIN} due to integration constants or, for example, more dynamically, due to the presence of massive flavors. Then define

$$u = 2f - 2g \qquad\qquad \Rightarrow \qquad\qquad \dot{u} = 6(1 - e^u) + \frac{1}{\rho}\,, \tag{2.41}$$

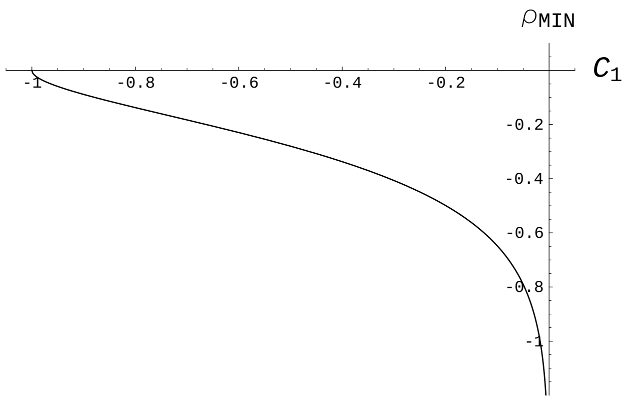
whose solution is

$$e^u = \frac{-6\rho\, e^{6\rho}}{(1 - 6\rho)e^{6\rho} + c_1}\,. \tag{2.42}$$

The constant of integration c_1 cannot be reabsorbed, and according to its value the solution dramatically changes in the IR. A systematic analysis of the various behaviors is presented in section 2.4. The value of c_1 determines whether there is a (negative) minimum value for the radial coordinate ρ. The requirement that the function e^u be positive defines three cases:

$$\begin{aligned} -1 < c_1 < 0 \qquad &\rightarrow \qquad \rho_{\mathrm{MIN}} \le \rho \le 0 \\ c_1 = 0 \qquad &\rightarrow \qquad -\infty < \rho \le 0 \\ c_1 > 0 \qquad &\rightarrow \qquad -\infty < \rho \le 0\,. \end{aligned}$$

In the case $-1 < c_1 < 0$, the minimum value ρ_{MIN} is given by an implicit equation. It can be useful to plot this value as a function of c_1:



$$0 = (1 - 6\rho_{\mathrm{MIN}})\, e^{6\rho_{\mathrm{MIN}}} + c_1$$

As it is clear from the graph, as $c_1 \to -1^+$ the range of the solution in ρ between the IR and the UV Landau pole shrinks to zero size, while in the limit $c_1 \to 0^-$ we no longer have a minimum radius.

The functions $g(\rho)$ and $f(\rho)$ can be analytically integrated, while the warp factor $h(\rho)$ and the original radial coordinate $r(\rho)$ cannot (in the particular case $c_1 = 0$ we found an explicit expression for the warp factor). By absorbing an irrelevant integration constant into a rescaling of r and $x^{0,1,2,3}$, we get:

$$e^g = \Big[(1-6\rho)e^{6\rho}+c_1\Big]^{1/6} \tag{2.43}$$

$$e^f = \sqrt{-6\rho}\, e^{3\rho}\Big[(1-6\rho)e^{6\rho}+c_1\Big]^{-1/3} \tag{2.44}$$

$$h(\rho) = -27\pi N_c \int_0^\rho e^{-4g} + c_2 \tag{2.45}$$

$$r(\rho) = \int^\rho e^f\ . \tag{2.46}$$

This solution is a very important result of our paper. We accomplished in finding a supergravity solution describing a (large) N_f number of backreacting D7-branes, smeared on the background produced by D3-branes at the tip of a conifold geometry.

The constant c_1 and c_2 correspond in field theory to switching on VEV's for relevant operators, as we will see in section 2.5.3. Moreover, in the new radial coordinate ρ, the metric reads

$$ds^2 = h^{-\frac12}dx_{1,3}^2 + h^{\frac12}e^{2f}\Big\{d\rho^2 + \frac{e^{2g-2f}}{6}\sum_{i=1,2}(d\theta_i^2+\sin^2\theta_i\, d\varphi_i^2) + \frac19(d\psi + \sum_{i=1,2}\cos\theta_i\, d\varphi_i)^2\Big\}\ . \tag{2.47}$$

2.4 Analysis of the solution: asymptotics and singularities

We perform here a systematic analysis of the possible solutions of the BPS system, and study the asymptotics in the IR and in the UV. In this section we will make use of the following formula for the Ricci scalar curvature, which can be obtained for solutions of the BPS system:

$$R = -2\,\frac{3N_f}{4\pi}\, h^{-1/2}e^{-2g+\frac12\phi}\left[7 + 4\frac{3N_f}{4\pi}e^{2g-2f+\phi}\right]. \tag{2.48}$$

2.4.1 The solution with $c_1 = 0$

Although the warp factor $h(\rho)$ cannot be analytically integrated in general, it can be if the integration constant c_1 is equal to 0. Indeed, introducing the *incomplete gamma function*, defined as follows:

$$\Gamma[a,x] \equiv \int_x^\infty t^{a-1}e^{-t}dt \xrightarrow[x\to-\infty]{} e^{i2\pi a}e^{-x}\left(\frac1x\right)^{1-a}\Big\{1+\mathcal{O}\Big(\frac1x\Big)\Big\}\,, \tag{2.49}$$

we can integrate

$$\begin{aligned} h(\rho) &= -27\pi N_c\int d\rho\frac{e^{-4\rho}}{(1-6\rho)^{2/3}} + c_2 \\ &= \frac92\pi N_c(\frac{3}{2e^2})^{1/3}\Gamma[\frac13, -\frac23+4\rho] + c_2 \\ &\simeq \frac{27}{4}\pi N_c(-6\rho)^{-2/3}e^{-4\rho} \ \text{ for }\ \rho\to-\infty\,. \end{aligned} \tag{2.50}$$

The warp factor diverges for $\rho \to -\infty$, and the integration constant c_2 disappears in the IR. Moreover, if we integrate the proper line element ds from a finite point to $\rho = -\infty$, we see that the throat has an *infinite invariant length*.

The function $r(\rho)$ cannot be given as an analytic integral, but using the asymptotic behavior of e^f for $\rho \to -\infty$ we can approximately integrate it:

$$r(\rho) \simeq 6^{1/6}\Big[(-\rho)^{1/6}e^{\rho} + \frac{1}{6}\Gamma[\frac{1}{6}, -\rho]\Big] + c_3 \tag{2.51}$$

in the IR. Fixing $r \to 0$ when $\rho \to -\infty$ we set $c_3 = 0$. We approximate further on

$$r(\rho) \simeq (-6\rho)^{1/6}e^{\rho} \; . \tag{2.52}$$

Substituting r in the asymptotic behavior of the functions appearing in the metric, we find that up to logarithmic corrections of relative order $1/|\log(r)|$:

$$\begin{aligned} e^{g(r)} &\simeq e^{f(r)} \simeq r \\ h(r) &\simeq \frac{27\pi N_c}{4}\frac{1}{r^4} \; . \end{aligned} \tag{2.53}$$

Therefore the geometry approaches $AdS_5 \times T^{1,1}$ with logarithmic corrections in the IR limit $\rho \to -\infty$.

2.4.2 UV limit

The solutions with backreacting flavors have a Landau pole in the ultraviolet ($\rho \to 0^-$), since the dilaton diverges (see (2.40)). The asymptotic behaviors of the functions appearing in the metric are:

$$e^{2g} \simeq (1+c_1)^{1/3}\Big[1 - \frac{6\rho^2}{1+c_1} + \mathcal{O}(\rho^3)\Big] \tag{2.54}$$

$$e^{2f} \simeq -6\rho\,(1+c_1)^{-2/3}\Big[1 + 6\rho + \mathcal{O}(\rho^2)\Big] \tag{2.55}$$

$$h \simeq c_2 + 27\pi N_c(1+c_1)^{-2/3}\Big[-\rho - \frac{4}{1+c_1}\rho^3 + \mathcal{O}(\rho^4)\Big] \; . \tag{2.56}$$

Note that we have used (2.45) for the warp factor. One concludes that $h(\rho)$ is monotonically decreasing with ρ; if it is positive at some radius, then it is positive down to the IR. If the integration constant c_2 is larger than zero, h is always positive and approaches c_2 at the Landau pole (UV). If $c_2 = 0$, then h goes to zero at the pole. If c_2 is negative, then the warp factor vanishes at $\rho_{\mathrm{MAX}} < 0$ before reaching the pole (and the curvature diverges there). The physically relevant solutions seem to have $c_2 > 0$.

The curvature invariants, evaluated in string frame, diverge when $\rho \to 0^-$, indicating that the supergravity description cannot be trusted in the UV. For instance the Ricci scalar $R \sim (-\rho)^{-5/2}$ if $c_2 \neq 0$, whereas $R \sim (-\rho)^{-3}$ if $c_2 = 0$. If $c_2 < 0$, then the Ricci scalar $R \sim (\rho_{\mathrm{MAX}} - \rho)^{-1/2}$ when $\rho \to \rho_{\mathrm{MAX}}^-$.

JHEP02(2007)090

2.4.3 IR limit

The IR ($\rho \to -\infty$) limit of the geometry of the flavored solutions is independent of the number of flavors, if we neglect logarithmic corrections to the leading term. Indeed, at the leading order, flavors decouple from the theory in the IR (see the discussion below eq. (2.5)). The counterpart in our supergravity plus branes solution is evident when we look at the BPS system (2.20): when $\rho \to -\infty$ the e^ϕ term disappears from the system, together with all the backreaction effects of the D7-branes (see appendix C for a detailed analysis of this phenomena), therefore the system reduces to the unflavored one.

- $c_1 = 0$
  The asymptotics of the functions appearing in the metric in the IR limit $\rho \to -\infty$ are:

$$e^g \simeq e^f \simeq (-6\rho)^{1/6} e^\rho \tag{2.57}$$

$$h \simeq \frac{27}{4}\pi N_c (-6\rho)^{-2/3} e^{-4\rho} \,. \tag{2.58}$$

  Formula (2.48) implies that the scalar curvature in string frame vanishes in the IR limit: $R^{(S)} \sim (-\rho)^{-1/2} \to 0$. An analogous but lengthier formula for the square of the Ricci tensor gives

$$R^{(S)}_{MN} R^{(S)\,MN} = \frac{160}{9\pi^2} \frac{N_f}{N_c}(-\rho) + \mathcal{O}(1) \quad \to \quad \infty \,, \tag{2.59}$$

  thus the supergravity description presents a singularity and some care is needed when computing observables from it. The same quantities in Einstein frame have limiting behavior $R^{(E)} \sim (-\rho)^{-1/2} \to 0$ and $R^{(E)}_{MN} R^{(E)\,MN} \to 640/(27\pi N_c)$.

- $c_1 > 0$
  The asymptotics in the limit $\rho \to -\infty$ are:

$$e^g \simeq c_1^{1/6} \tag{2.60}$$

$$e^f \simeq c_1^{-1/3} (-6\rho)^{1/2} e^{3\rho} \tag{2.61}$$

$$h \simeq 27\pi N_c c_1^{-2/3} (-\rho) \,. \tag{2.62}$$

  Although the radial coordinate ranges down to $-\infty$, the throat has a *finite invariant length.* The Ricci scalar in string frame is $R \sim (-\rho)^{-3} e^{-6\rho} \to -\infty$.

- $c_1 < 0$
  In this case the IR limit is $\rho \to \rho_{\rm MIN}$. The asymptotics in this limit are:

$$e^g \simeq \big(-6\rho_{\rm MIN} e^{6\rho_{\rm MIN}} \big)^{1/6} (6\rho - 6\rho_{\rm MIN})^{1/6} \tag{2.63}$$

$$e^f \simeq \big(-6\rho_{\rm MIN} e^{6\rho_{\rm MIN}} \big)^{1/6} (6\rho - 6\rho_{\rm MIN})^{-1/3} \tag{2.64}$$

$$h \simeq const. > 0 \,. \tag{2.65}$$

  The throat has a *finite invariant length.* The Ricci scalar in string frame is $R \sim (\rho - \rho_{\rm MIN})^{-1/3} \to \infty$.

Using the criterion in [37], that proposes the IR singularity to be physically acceptable if g_{tt} is bounded near the IR problematic point, we observe that these singular geometries are all acceptable. Gauge theory physics can be read from these supergravity backgrounds. We call them “good singularities”.

2.5 Detailed study of the dual field theory

In this section we are going to undertake a detailed analysis of the dual gauge theory features, reproduced by the supergravity solution. The first issue we want to address is what is the effect of the smearing on the gauge theory dual.

As we wrote above, the addition to the supergravity solution of one stack of localized noncompact D7-branes at $z_1 = 0$ put in the field theory flavors coupled through a superpotential term

$$W = \lambda \operatorname{Tr}(A_i B_k A_j B_l)\, \epsilon^{ij} \epsilon^{kl} + h_1\, \tilde{q}^a A_1 Q_a + h_2\, \tilde{Q}^a B_1 q_a \ , \tag{2.66}$$

where we explicitly wrote the flavor indices a. For this particular embedding the two branches are localized, say, at $\theta_1 = 0$ and $\theta_2 = 0$ respectively on the two spheres. One can exhibit a lot of features in common with the supergravity plus D7-branes solution:

- the theory has $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ flavor symmetry (the diagonal axial $\mathrm{U}(1)_A$ is anomalous), each group corresponding to one branch of D7's;
- putting only one branch there are gauge anomalies in QFT and a tadpole in SUGRA, while for two branches they cancel;
- adding a mass term for the fundamentals the flavor symmetry is broken to the diagonal $\mathrm{U}(N_f)$, while in SUGRA there are embeddings moved away from the origin for which the two branches merge.

The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ part of the isometry group of the background without D7's is broken by the presence of localized branes. It amounts to separate rotations of the two S^2 in the geometry and shifts the location of the branches. Its action is realized through the superpotential, and exploiting its action we can obtain the superpotential for D7-branes localized in other places. The two bifundamental doublets A_j and B_j transform as spinors of the respective SU(2). So the flavor superpotential term for a configuration in which the two branches are located at x and y on the two spheres can be obtained by identifying two rotations that bring the north pole to x and y. There is of course a $\mathrm{U}(1) \times \mathrm{U}(1)$ ambiguity in this. Then we have to act with the corresponding SU(2) matrices U_x and U_y on the vectors (A_1, A_2) and (B_1, B_2) (which transform in the $(\mathbf{2}, 1)$ and $(1, \mathbf{2})$ representations) respectively, and select the first vector component. In summary we can write [8]

$$W_f = h_1\, \tilde{q}^x \Big[U_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \Big]_1 Q_x + h_2\, \tilde{Q}^y \Big[U_y \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Big]_1 q_y \ , \tag{2.67}$$

[8] In case the two gauge couplings and theta angles are equal, we could appeal to the $\mathbb{Z}_2$ symmetry that exchanges them to argue $|h_1| = |h_2|$, but no more because of the ambiguities.

where the notation $\tilde{q}^x$, Q_x stands for the flavors coming from a first D7 branch being at x, and the same for a second D7 branch at y.

To understand the fate of the two phase ambiguities in the couplings h_1 and h_2, we appeal to symmetries. The $\mathrm{U}(1)$ action which gives $(q, \tilde{q}, Q, \tilde{Q})$ charges $(1, -1, -1, 1)$ is a symmetry explicitly broken by the flavor superpotential. The freedom of redefining the flavor fields acting with this $\mathrm{U}(1)$ can be exploited to reduce to the case in which the phase of the two holomorphic couplings is the same. The $\mathrm{U}(1)$ action with charges $(1, 1, 1, 1)$ is anomalous with equal anomalies for both the gauge groups, and it can be used to absorb the phase ambiguity into a shift of the sum of Yang-Mills theta angles $\theta_1^{\mathrm{YM}} + \theta_2^{\mathrm{YM}}$ (while the difference holds steady). This is what happens for D7-branes on flat spacetime. The ambiguity we mentioned amounts to rotations of the transverse $\mathbb{R}^2$ space, whose only effect is a shift of C_0. As we show in the next section, the value of C_0 is our way of measuring the sum of theta angles through probe D(-1)-branes. Notice that if we put in our setup many separate stacks of D7's, all their superpotential $\mathrm{U}(1)$ ambiguities can be reabsorbed in a single shift of C_0.

From a physical point of view, the smearing corresponds to put the D7-branes at different points on the two spheres, distributing each branch on one of the 2-spheres. This is done homogeneously so that there is one D7 at every point of S^2. The non-anomalous flavor symmetry is broken from $\mathrm{U}(1)_B \times \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L$ (localized configuration) to $\mathrm{U}(1)_B \times \mathrm{U}(1)_V^{N_f-1} \times \mathrm{U}(1)_A^{N_f-1}$ (smeared configuration).[9]

Let us introduce a pair of flavor indices (x, y) that naturally live on $S^2 \times S^2$ and specify the D7. The superpotential for the whole system of smeared D7-branes is just the sum (actually an integral) over the indices (x, y) of the previous contributions:

$$W = \lambda \,\mathrm{Tr}(A_i B_k A_j B_l)\, \epsilon^{ij} \epsilon^{kl} + h_1 \int_{S^2} d^2x\, \tilde{q}^x \Big[U_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \Big]_1 Q_x + h_2 \int_{S^2} d^2y\, \tilde{Q}^y \Big[U_y \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Big]_1 q_y \,. \tag{2.68}$$

Again, all the $\mathrm{U}(1)$ ambiguities have been reabsorbed in field redefinitions and a global shift of $\theta_1^{\mathrm{YM}} + \theta_2^{\mathrm{YM}}$.

In this expression the $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ symmetry is manifest: rotations of the bulk fields A_j, B_j leave the superpotential invariant because they can be reabsorbed in rotations of the dummy indices (x, y). In fact, the action of $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ on the flavors is a subgroup of the broken $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ flavor symmetry. In the smeared configuration, there is a D7-brane at each point of the spheres and the group $\mathrm{SU}(2)^2$ rotates all the D7's in a rigid way, moving each D7 where another was. So it is a flavor transformation contained in $\mathrm{U}(N_f)^2$. By combining this action with a rotation of A_i and B_i, we get precisely the claimed symmetry.

Even if written in an involved fashion, the superpotential (2.68) does not spoil the features of the gauge theory. In particular, the addition of a flavor mass term still would

[9]The axial $\mathrm{U}(1)$ which gives charges $(1, 1, -1, -1)$ to one set of fields $(q_x, \tilde{q}^x, Q_x, \tilde{Q}^x)$ coming from a single D7, is an anomalous symmetry. For every D7-brane we consider, the anomaly amounts to a shift of the same two theta angles of the gauge theory. So we can combine this $\mathrm{U}(1)$ with an axial rotation of all the flavor fields, and get an anomaly free symmetry. In total, from N_f D7's we can find $N_f - 1$ such anomaly free axial $\mathrm{U}(1)$ symmetries.

JHEP02(2007)090

give rise to the symmetry breaking pattern

$$\mathrm{U}(1)_B \times \mathrm{U}(1)_V^{N_f-1} \times \mathrm{U}(1)_A^{N_f-1} \quad \rightarrow \quad \mathrm{U}(1)_V^{N_f} \ .$$

2.5.1 Holomorphic gauge couplings and β-functions

In order to extract information on the gauge theory from the supergravity solution, we need to know the holographic relations between the gauge couplings, the theta angles and the supergravity fields. These formulae can be properly derived only in the orbifold $\mathbb{R}^{1,3} \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$, where string theory can be quantized, by considering fractional branes placed at the singularity. The near-horizon geometry describing the IR dynamics on a stack of N regular branes at the singularity is $AdS_5 \times S^5/\mathbb{Z}_2$. The dual gauge theory is an $\mathcal{N} = 2$ $\mathrm{SU}(N) \times \mathrm{SU}(N)$ SCFT with bifundamental hypermultiplets. In $\mathcal{N} = 1$ language, an $\mathcal{N} = 2$ vector multiplet decomposes into a vector multiplet and a chiral multiplet in the adjoint of the gauge group, whereas a bifundamental hypermultiplet decomposes into two bifundamental chiral multiplets. Klebanov and Witten [6] recognized that giving equal (but opposite) complex mass parameters to the adjoint chiral superfields of this $\mathcal{N} = 2$ SCFT, an RG flow starts whose IR fixed point is described by the gauge theory dual to the $AdS_5 \times T^{1,1}$ geometry.

In the $\mathcal{N} = 2$ orbifold theory, the holographic relations can be derived exactly. The result is the following:

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi e^{-\phi}}{g_s} \tag{2.69}$$

$$\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} = \frac{e^{-\phi}}{g_s}\left[\frac{1}{2\pi\alpha'}\int_{S_2} B_2 - \pi \pmod{2\pi}\right] \tag{2.70}$$

$$\theta_1^{\mathrm{YM}} = \pi C_0 + \frac{1}{2\pi}\int_{S_2} C_2 \pmod{2\pi} \tag{2.71}$$

$$\theta_2^{\mathrm{YM}} = \pi C_0 - \frac{1}{2\pi}\int_{S_2} C_2 \pmod{2\pi} \tag{2.72}$$

where the integrals are performed over the 2-sphere that shrinks at the orbifold fixed point and could be blown-up. The ambiguity in (2.70) is the 2π periodicity of $\frac{1}{2\pi\alpha'}\int_{S_2} B_2$ which comes from the quantization condition on H_3 (if fractional branes are absent). A shift of 2π amounts to move to a dual description of the gauge theory.[10] The ambiguities of RR fields are more subtle: the periodicities in (2.71) and (2.72) correspond to the two kinds of fractional D(-1)-branes appearing in the theory. The angles θ_1^{YM} and θ_2^{YM} come from the imaginary parts of the action of the two kinds of fractional Euclidean D(-1) branes. Both of them are then defined *modulo* 2π in the quantum field theory:

$$(\theta_1^{\mathrm{YM}}\,,\,\theta_2^{\mathrm{YM}}) \equiv (\theta_1^{\mathrm{YM}} + 2\pi\,,\,\theta_2^{\mathrm{YM}}) \equiv (\theta_1^{\mathrm{YM}}\,,\,\theta_2^{\mathrm{YM}} + 2\pi)\ . \tag{2.73}$$

[10] In the KW theory, this is Seiberg duality. Notice that the periodicity must fail once flavor fields are added.

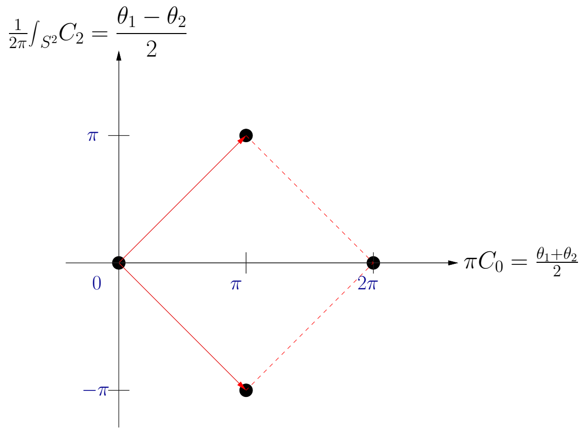


Figure 3: Unit cell of the lattice of Yang-Mills θ angles and RR fields integrals.

On the string theory side the periodicities exactly match: an Euclidean fractional D(-1)-brane enters the functional integral with a term $\exp\{-\frac{8\pi^2}{g_j^2} + i\theta_j^{\rm YM}\}$.[11] Hence the imaginary part in the exponent is defined *modulo* 2π in the quantum string theory. The identification (2.73) of the field theory translates on the string side in:

$$(\pi C_0\,,\ \frac{1}{2\pi}\int_{S^2} C_2) \equiv (\pi C_0 + \pi\,,\ \frac{1}{2\pi}\int_{S^2} C_2 + \pi) \equiv (\pi C_0 + \pi\,,\ \frac{1}{2\pi}\int_{S^2} C_2 - \pi)\ . \tag{2.74}$$

The lattice is shown in figure 3. The vectors of the unit cell drawn in the figure are the ones defined by fractional branes.

From figure 3 and (2.74) we can see that:

$$\pi C_0 \equiv \pi C_0 + 2\pi\ . \tag{2.75}$$

This is indeed the identification that arises from considering a regular D(-1) brane, which can be seen as a linear superposition of the two kinds of fractional D(-1)-branes. Notice that the closed string field C_0 in this orbifold has periodicity 2, differently from the periodicity 1 in flat space. This is due to the fact that in the orbifold the fundamental physical objects are the fractional branes.

Usually in the literature the afore-mentioned holographic relations were assumed to hold also in the conifold case. Strassler remarked in [13] that for the conifold theory the formulae for the sum of the gauge couplings and the sum of theta angles need to be corrected. We expect that the formula for the sum of theta angles is correct as far as anomalies are concerned, since anomalies do not change in RG flows. Instead the formula (2.69) may need to be corrected in the KW theory: in general the dilaton could be identified with some combination of the gauge and superpotential couplings.

Let us now make contact with our supergravity solution. In the smeared solution, since $dF_1 \neq 0$ at every point, it is not possible to define a scalar potential C_0 such that

[11] We have written the complexified gauge coupling instead of the supergravity fields for the sake of brevity: the use of the dictionary is understood.

$F_1 = dC_0$. We by-pass this problem by restricting our attention to the non-compact 4-cycle defined by $\{\rho, \psi, \theta_1 = \theta_2, \varphi_1 = 2\pi - \varphi_2\}$ [41](note that it wraps the R-symmetry direction ψ), so that we can pull-back on it and write

$$F_1^{\text{eff}} = \frac{N_f}{4\pi}\, d\psi \tag{2.76}$$

and therefore

$$C_0^{\text{eff}} = \frac{N_f}{4\pi}(\psi - \psi_0)\ . \tag{2.77}$$

Now we can identify:

$$\frac{8\pi^2}{g^2} = \pi\, e^{-\phi} = -\frac{3N_f}{4}\rho \tag{2.78}$$

$$\theta_1^{\text{YM}} + \theta_2^{\text{YM}} = \frac{N_f}{2}(\psi - \psi_0)\ , \tag{2.79}$$

where we suppose for simplicity the two gauge couplings to be equal ($g_1 = g_2 \equiv g$). The generalization to an arbitrary constant B_2 is straightforward since the difference of the inverse squared gauge couplings does not run. Although, as discussed above, one cannot be sure of the validity of (2.78), we can try to extract some information.

Let us first compute the β-function of the gauge couplings. The identification (2.69) allows us to define a "radial" β-function that we can directly compute from supergravity [38]:

$$\beta^{(\rho)}_{\frac{8\pi^2}{g^2}} \equiv \frac{\partial}{\partial\rho}\frac{8\pi^2}{g^2} = \pi\frac{\partial e^{-\phi}}{\partial\rho} = -\frac{3N_f}{4}\ . \tag{2.80}$$

(Compare this result with eq. (2.5)). The physical β-function defined in the field theory is of course:

$$\beta_{\frac{8\pi^2}{g^2}} \equiv \frac{\partial}{\partial\log\frac{\mu}{\Lambda}}\frac{8\pi^2}{g^2}\ , \tag{2.81}$$

where μ is the subtraction scale and Λ is a renormalization group invariant scale. In order to get the precise field theory β-function from the supergravity computation one needs the *energy-radius* relation $\rho = \rho\big(\frac{\mu}{\Lambda}\big)$, from which $\beta = \beta^{(\rho)}\ \partial\rho/\partial\log\frac{\mu}{\Lambda}$. In general, for non-conformal duals, the radius-energy relation depends on the phenomenon one is interested in and accounts for the scheme-dependence in the field theory.

Even without knowing the radius-energy relation, there is some physical information that we can extract from the radial β-function (2.80). In particular, being the energy-radius relation $\rho = \rho\big(\frac{\mu}{\Lambda}\big)$ monotonically increasing, the signs of the two beta functions coincide.

In our case, using $r = \frac{\mu}{\Lambda}$ and eq. (2.52), one gets matching between (2.5) and (2.80).

2.5.2 R-symmetry anomaly and vacua

Now we move to the computation of the $\mathrm{U}(1)_R$ anomaly. On the field theory side we follow the convention that the R-charge of the superspace Grassmann coordinates is $R[\vartheta] = 1$. This fixes the R-charge of the gauginos $R[\lambda] = 1$. Let us consider an infinitesimal R-symmetry transformation and calculate the $\mathrm{U}(1)_R - \mathrm{SU}(N_c) - \mathrm{SU}(N_c)$ triangle anomaly. The anomaly coefficient in front of the instanton density of a gauge group is $\sum_f R_f T[\mathcal{R}^{(f)}]$, where the sum runs over the fermions f, R_f is the R-charge of the fermion and $T[\mathcal{R}^{(f)}]$ is the Dynkin index of the gauge group representation $\mathcal{R}^{(f)}$ the fermion belongs to, normalized as $T[\mathcal{R}^{(fund.)}] = 1$ and $T[\mathcal{R}^{(adj.)}] = 2N_c$. Consequently the anomaly relation in our theory is the following:

$$\partial_\mu J^\mu_R = -\frac{N_f}{2}\frac{1}{32\pi^2}\left(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a + G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a\right) , \tag{2.82}$$

or in other words, under a $\mathrm{U}(1)_R$ transformation of parameter ε, for both gauge groups the theta angles transform as

$$\theta^{\mathrm{YM}}_i \to \theta^{\mathrm{YM}}_i - \frac{N_f}{2}\varepsilon . \tag{2.83}$$

On the string/gravity side a $\mathrm{U}(1)_R$ transformation of parameter ε is realized (in our conventions) by the shift $\psi \to \psi - 2\varepsilon$. This can be derived from the transformation of the complex variables (2.1), which under a $\mathrm{U}(1)_R$ rotation get $z_i \to e^{-i\varepsilon}z_i$, or directly by the decomposition of the 10d spinor ϵ into 4d and 6d factors and the identification of the 4d anti-supercharge with the 4d spinor. By means of the dictionary (2.79) we obtain:

$$\theta^{\mathrm{YM}}_1 + \theta^{\mathrm{YM}}_2 \to \theta^{\mathrm{YM}}_1 + \theta^{\mathrm{YM}}_2 - 2\,\frac{N_f}{2}\varepsilon , \tag{2.84}$$

in perfect agreement with (2.83).

The $\mathrm{U}(1)_R$ anomaly is responsible for the breaking of the symmetry group, but usually a discrete subgroup survives. Disjoint physically equivalent vacua, not connected by other continuous symmetries, can be distinguished thanks to the formation of domain walls among them, whose tension could also be measured. We want to read the discrete symmetry subgroup of $\mathrm{U}(1)_R$ and the number of vacua both from field theory and supergravity. In field theory the $\mathrm{U}(1)_R$ action has an extended periodicity (range of inequivalent parameters) $\varepsilon \in [0, 8\pi)$ instead of the usual 2π periodicity, because the minimal charge is 1/4. Let us remark however that when ε is a multiple of 2π the transformation is not an R-symmetry, since it commutes with supersymmetry. The global symmetry group contains the baryonic symmetry $\mathrm{U}(1)_B$ as well, whose parameter we call $\alpha \in [0, 2\pi)$, and the two actions $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_B$ satisfy the following relation: $\mathcal{U}_R(4\pi) = \mathcal{U}_B(\pi)$. Therefore the group manifold $\mathrm{U}(1)_R \times \mathrm{U}(1)_B$ is parameterized by $\varepsilon \in [0, 4\pi)$, $\alpha \in [0, 2\pi)$ (this parameterization realizes a nontrivial torus) and $\mathrm{U}(1)_B$ is a true symmetry of the theory. The theta angle shift (2.83) allows us to conclude that the $\mathrm{U}(1)_R$ anomaly breaks the symmetry according to $\mathrm{U}(1)_R \times \mathrm{U}(1)_B \to \mathbb{Z}_{N_f} \times \mathrm{U}(1)_B$, where the latter is given by $\varepsilon = 4n\pi/N_f$ $(n = 0, 1, \ldots, N_f - 1)$, $\alpha \in [0, 2\pi)$.

Coming to the string side, the solution for the metric, the dilaton and the field strengths is invariant under arbitrary shifts of ψ. But the nontrivial profile of C_0, which can be probed by D(-1)-branes for instance, breaks this symmetry. The presence of DBI actions in the functional integral tells us that the RR potentials are quantized, in particular C_0 is defined modulo integers. Taking the formula (2.77) and using the periodicity 4π of ψ, we conclude that the true invariance of the solution is indeed $\mathbb{Z}_{N_f}$.

One can be interested in computing the domain wall tension in the field theory by means of its dual description in terms of a D5-brane with 3 directions wrapped on a

3-sphere (see [39] for a review in the conifold geometry). It is easy to see that, as in Klebanov-Witten theory, this object is stable only at $r = 0$ ($\rho \to -\infty$), where the domain wall is tensionless.

2.5.3 The UV and IR behaviors

The supergravity solution allows us to extract the IR dynamics of the KW field theory with massless flavors. Really what we obtained is a class of solutions, parameterized by two integration constants c_1 and c_2. Momentarily, we will say something about their meaning but anyway some properties are independent of them.

The fact that the β-function is always positive, with the only critical point at vanishing gauge coupling, tell us that the theory is irreparably driven to that point, unless the supergravity approximation breaks down before ($c_1 < 0$), for instance because of the presence of curvature singularities. Using the ρ coordinate this is clear-cut. In cases where the string coupling falls to zero in the IR, the gravitational coupling of the D7 to the bulk fields also goes to zero and the branes tend to decouple. The signature of this is in equation (2.37) of the BPS system: the quantity $e^{\phi} N_f$ can be thought of as the effective size of the flavor backreaction which indeed vanish in the far IR. The upshot is that flavors can be considered as an "irrelevant deformation" of the $AdS_5 \times T^{1,1}$ geometry.

The usual technique for studying deformations of an AdS_5 geometry is through the GKPW [2, 3] formula in AdS/CFT. Looking at the asymptotic behavior of fields in the AdS_5 effective theory:[12]

$$\delta\Phi = a\, r^{\Delta-4} + c\, r^{-\Delta} \,, \tag{2.85}$$

we read, on the CFT side, that the deformation is $H = H_{\rm CFT} + a\,\mathcal{O}$ with $c = \langle \mathcal{O} \rangle$ the VEV of the operator corresponding to the field Φ, and Δ the quantum dimension of the operator $\mathcal{O}$. Alternatively, one can compute the effective 5d action and look for the masses of the fields, from which the dimension is extracted with the formula:

$$\Delta = 2 + \sqrt{4 + m^2} \,, \tag{2.86}$$

with the mass expressed in units of inverse AdS radius. We computed the 5d effective action for the particular deformations $e^{f(r)}$, $e^{g(r)}$ and $\phi(r)$ and including the D7-brane action terms (the details are in section 3). After diagonalization of the effective Kähler potential, we got a scalar potential V containing a lot of information. First of all, minima of V corresponds to the AdS_5 geometries, that is conformal points in field theory. The only minimum is formally at $e^{\phi} = 0$, and has the $AdS_5 \times T^{1,1}$ geometry. Then, expanding the potential at quadratic order the masses of the fields can be read; from here we deduce that we have operators of dimension 6 and 8 taking VEV, and a marginally irrelevant operator inserted.[13]

[12] Notice that usually the GKPW prescription or the holographic renormalization methods are used when we may have flows starting from a conformal point in the UV. In this case, our conformal point is in the IR and one may doubt about the validity in this unconventional case. See section 6 in the paper [40] for an indication that applying the prescription in an IR point makes sense, even when the UV geometry is very far away from $AdS_5 \times M_5$. We thank Kostas Skenderis for correspondence on this issue.

[13] To distinguish between a VEV and an insertion we have to appeal to the first criterium described in eq. (2.85) and below.

The operators taking VEV where already identified in [8, 35]. The dimension 8 operator is $\mathrm{Tr}F^4$ and represents the deformation from the conformal KW solution to the non-conformal 3-brane solution. The dimension 6 operator is a combination of the operators $\mathrm{Tr}(\mathcal{W}_\alpha \bar{\mathcal{W}}^\alpha)^2$ and represents a relative metric deformation between the $S^2 \times S^2$ base and the U(1) fiber of $T^{1,1}$. The marginally irrelevant insertion is the flavor superpotential, which would be marginal at the hypothetic AdS_5 (conformal) point with $e^\phi = 0$, but is in fact irrelevant driving the gauge coupling to zero in the IR and to very large values in the UV. Let us add that the scalar potential V can be derived from a superpotential W, from which in turn the BPS system (2.20) can be obtained.

Since in the IR the flavor branes undergo a sort of decoupling, the relevant deformations dominate and their treatment is much the same as for the unflavored Klebanov-Witten solution [8, 35, 13]. We are not going to repeat it here, and we will concentrate on the case $c_1 = c_2 = 0$. The supergravity solution flows in the IR to the $AdS_5 \times T^{1,1}$ solution (with corrections of relative order $1/|\log(r)|$). On one hand the R-charges and the anomalous dimensions tend to the almost conformal values:

$$\begin{aligned} R_{A,B} &= \frac{1}{2} \qquad\qquad & \gamma_{A,B} &= -\frac{1}{2} \\ R_{q,Q} &= \frac{3}{4} \qquad\qquad & \gamma_{q,Q} &= \frac{1}{4} \,. \end{aligned} \tag{2.87}$$

Using the formula for the β-function of a superpotential dimensionless coupling:

$$\beta_{\tilde{h}} = \tilde{h}\Big[-3 + \sum\nolimits_\Phi \big(1 + \frac{\gamma_\Phi}{2}\big)\Big] \,, \tag{2.88}$$

where Φ are the fields appearing in the superpotential term, we obtain that the total superpotential (2.68) is indeed marginal. On the other hand the gauge coupling flows to zero. Being at an almost conformal point, we can derive the radius-energy relation through rescalings of the radial and Minkowski direction, getting $r = \mu/\Lambda$. Then the supergravity beta function coincides with the exact (perturbative) holomorphic β-function (in the Wilsonian scheme):[14]

$$\beta_g = -\frac{g^3}{16\pi^2}\Big[3N_c - 2N_c(1-\gamma_A) - N_f(1-\gamma_f)\Big] \,. \tag{2.89}$$

If we are allowed to trust the orbifold relation (2.69) relating gauge coupling constants and dilaton, we conclude that the gauge coupling flows to zero in the IR. This fact could perhaps explain the divergence of the curvature invariants in string frame [4], as revealed by (2.59). The field theory would enter the perturbative regime at this point. However, it is hard to understand why the anomalous dimensions of the fields are large while the theory seems to become perturbative. For this reason, we question the validity in the conifold case of the holographic relation (2.69), that can be derived only for the orbifold.

[14] Here it is manifest why the SUGRA β-function computed in this context with probe branes matches the field theory one, even if this requires the absence of order N_f/N_c corrections to the anomalous dimensions $\gamma_{A,B}$, which one does not know how to derive (the stress-energy tensor is linear in N_f/N_c). It is because those corrections are really of order $e^\phi N_f/N_c$, and in the IR $e^\phi \to 0$.

JHEP02(2007)090

In appendix C we propose an alternative interpretation of the IR regime of our field theory, based on some nice observations made in [13] about the KW field theory. We argue that the theory may flow to a strongly coupled fixed point, although the string frame curvature invariant is large, as in the Klebanov-Witten solution for small values of $g_s N_c$.

Contrary to the IR limit, the UV regime of the theory is dominated by flavors and we find the same kind of behavior for all values of the relevant deformations c_1 and c_2. The gauge couplings increase with the energy, irrespective of the number of flavors. At a finite energy scale that we conventionally fixed to $\rho = 0$, the gauge theory develops a Landau pole, as told by the string coupling that diverges at that particular radius. This energy scale is finite, because $\rho = 0$ is at finite proper distance from the bulk points $\rho < 0$.

At the Landau pole radius the supergravity description breaks down for many reasons: the string coupling diverges as well as the curvature invariants (both in Einstein and string frame), and the ψ circle shrinks. An UV completion must exist, and finding it is an interesting problem. One could think to obtain a new description in terms of supergravity plus branes through various dualities. In particular T-duality will map our solution to a system of NS5, D4 and D6-branes, which could then be uplifted to M-theory. Anyway, T-duality has to be applied with care because of the presence of D-branes on a non-trivial background, and we actually do not know how to T-dualize the Dirac-Born-Infeld action. We leave this interesting problem for the future.

3. Part II: generalizations

In this section we are going to extend the smearing procedure of the D7-brane, which was formulated in section 2 for the particular case of the $AdS_5 \times T^{1,1}$ space, to the more general case of a geometry of the type $AdS_5 \times M_5$, where M_5 is a five-dimensional compact manifold. Of course, the requirement of supersymmetry restricts greatly the form of M_5. Actually, we will verify that, when M_5 is Sasaki-Einstein, the formalism of section 2 can be easily generalized. As a result of this generalization we will get a more intrinsic formulation of the smearing, which eventually could be further generalized to other types of flavor branes in different geometries.

Following the line of thought that led to the action (1.7), let us assume that, for a general geometry, the effect of the smearing on the WZ term of the D7-brane action can be modelled by means of the substitution:

$$S_{WZ} = T_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 \;\rightarrow\; T_7 \int_{\mathcal{M}_{10}} \Omega \wedge C_8 \,, \tag{3.1}$$

where Ω is a two-form which determines the distribution of the RR charge of the D7-brane in the smearing and $\mathcal{M}_{10}$ is the full ten-dimensional manifold. For a supersymmetric brane one expects the charge density to be equal to the mass density and, thus, the smearing of the DBI part of the D7-brane action should be also determined by the form Ω. Let us explain in detail how this can be done. First of all, let us suppose that Ω is *decomposable*, i.e. that it can be written as the wedge product of two one-forms. In that case, at an arbitrary point, Ω would determine an eight-dimensional orthogonal hyperplane, which we are going to identify with the tangent space of the D7-brane worldvolume. A general two-form Ω will not be decomposable. However, it can be written as a finite sum of the type:

$$\Omega = \sum_i \Omega^{(i)} \,, \tag{3.2}$$

where each $\Omega^{(i)}$ is decomposable. At an arbitrary point, each of the $\Omega^{(i)}$'s is dual to an eight-dimensional hyperplane. Thus, Ω will determine locally a collection of eight-dimensional hyperplanes. In the smearing procedure, to each decomposable component of Ω we associate the volume form of its orthogonal complement in $\mathcal{M}_{10}$. Thus, the contribution of every $\Omega^{(i)}$ to the DBI action will be proportional to the ten-dimensional volume element. Accordingly, let us perform the following substitution:

$$S_{\rm DBI} = -T_7 \sum_{N_f} \int_{\mathcal{M}_8} d^8\xi \sqrt{-\hat{G}_8}\; e^{\phi} \quad \rightarrow \quad -T_7 \int_{\mathcal{M}_{10}} d^{10}x \sqrt{-G}\; e^{\phi} \sum_i \big|\,\Omega^{(i)}\,\big| \,, \tag{3.3}$$

where $\big|\,\Omega^{(i)}\,\big|$ is the modulus of $\Omega^{(i)}$ and represents the mass density of the i^{th} piece of Ω in the smearing. There is a natural definition of $\big|\,\Omega^{(i)}\,\big|$ which is invariant under coordinate transformations. Indeed, let us suppose that $\Omega^{(i)}$ is given by:

$$\Omega^{(i)} = \frac{1}{2!} \sum_{M,N} \Omega^{(i)}_{MN}\, dx^M \wedge dx^N \,. \tag{3.4}$$

Then, $\big|\,\Omega^{(i)}\,\big|$ is defined as follows:

$$\big|\,\Omega^{(i)}\,\big| \equiv \sqrt{\frac{1}{2!}\,\Omega^{(i)}_{MN}\,\Omega^{(i)}_{PQ}\, G^{MP}\, G^{NQ}} \,. \tag{3.5}$$

Notice that Ω acts as a magnetic source for the field strength F_1. Actually, from the equation of motion of C_8 one gets that Ω is just the violation of the Bianchi identity for F_1, namely:

$$dF_1 = \Omega \,. \tag{3.6}$$

For a supersymmetric configuration the form Ω is not arbitrary. Indeed, eq. (3.6) determines F_1 which, in turn, enters the equation that determines the Killing spinors of the background. On the other hand, Ω must come from the superposition (smearing) of κ-symmetric branes. When the manifold M_5 is Sasaki-Einstein, we will show in section 3.2 that Ω can be determined in terms of the Kähler form of the Kähler-Einstein base of M_5 and that the resulting DBI+WZ action is a direct generalization of the result written in (1.7). We will also show that the existence of Killing spinors implies that the functions appearing in the ansatz satisfy a system of first-order differential equations analogous to that written in (2.20).

3.1 General smearing and DBI action

Here we will elaborate on the previous construction: writing the DBI action for a general smearing of supersymmetric D7-branes. We mean that in general on an $\mathcal{N}=1$ background there is a continuous family of supersymmetric 4-manifolds[15] that D7-branes can wrap corresponding to quarks with the same mass and quantum numbers. All these configurations preserve the same four supercharges, so we can think of putting D7's arbitrarily distributed (with arbitrary density functions) on these manifolds. We want to write the DBI plus WZ action for this system.

Supersymmetry plays a key role. The fact that we can put D7's and not anti-D7's implies that the charge distribution completely specifies the system. For D7-branes the charge distribution is a 2-form Ω, which can be localized (a "delta form" or current) or smooth (for smeared systems). The Bianchi identity reads $dF_1 = \Omega$ and is easily implemented through the WZ action (3.1): $S_{\rm WZ} = T_7 \int \Omega \wedge C_8$. Notice that in this case a well defined Ω not only must be closed (which is charge conservation) but also exact. Supersymmetry also guides us in writing the DBI action, because the energy distribution must be equal to the charge distribution. But there is a subtlety here, because the energy distribution is not a 2-form, and some more careful analysis is needed.

Let us start considering the case of a single D7-brane localized on $\mathcal{M}_8$. We can write its DBI action as a bulk 10d integral by introducing a localized distribution 2-form Ω such that

$$\int_{\mathcal{M}_8} d^8\xi \, e^{\phi} \sqrt{-\hat{G}_8} = \int d^{10}x \, e^{\phi} \sqrt{-G} \, |\Omega| \, . \tag{3.7}$$

Ω is the Poincaré dual to $\mathcal{M}_8$. It can be (locally) written as $\Omega = \delta^{(2)}(\mathcal{M}_8)\sqrt{-\hat{G}_8}/\sqrt{-G}\,\alpha \wedge \beta$, through a properly normalized delta function and the product of two 1-forms (in general not separately globally defined) orthogonal to the 8-submanifold.[16] In particular it is decomposable.

The decomposability of a 2-form can be established through Plücker's relations, and the minimum number of decomposable pieces needed to write a general 2-form is half of its rank as a matrix.[17] So the decomposability of a 2-form at a point means that it is dual to one 8d hyperplane at that point; in general a 2-form is dual to a collection of 8d hyperplanes.

If we do a parallel smearing of our D7-brane we get a smooth charge distribution 2-form, non-zero at every point. This corresponds to put a lot of parallel D7's and go to the continuum limit. Being the smearing parallel, we never have intersections of branes and the 2-form is still decomposable. As a result (3.7) is still valid. If instead we construct a smeared system with intersection of branes, the charge distribution Ω is no longer decomposable. Every decomposable piece corresponds to one 8d hyperplane, tangent to one of the branes at the intersection. Since energy is additive, the DBI action is obtained by summing the moduli of the decomposable pieces (and not just taking the modulus of Ω). Every brane at the intersection defines its 8d hyperplane and gives its separate con-

[15]Even if we try to be general, we still stick to the case with vanishing B_2 background and vanishing F_{MN} on the brane world-volume.

[16]This orthogonality does not need a metric. A 1-form is a linear function from the tangent space to $\mathbb{R}$, and its kernel is a 9d hyperplane. The 8d hyperplane, tangent to the submanifold, orthogonal to the two 1-forms, is the intersection of the two kernels.

[17]The rank of an antisymmetric matrix is always even.

tribution to the DBI action and to the stress-energy tensor. We simply sum the separate contributions because of supersymmetry: the D7's do not interact among themselves due to the cancellation of attractive/repulsive forces. Notice that in doing the smearing of bent branes, one generically obtains unavoidable self-intersections.

Summarizing, given the splitting of the charge distribution 2-form into decomposable pieces $\Omega = \sum_k \Omega^{(k)}$, the DBI action reads

$$S_{\text{DBI}} = -T_7 \int d^{10}x \sqrt{-G}\, e^{\phi} \sum\nolimits_k \big|\Omega^{(k)}\big| \,. \tag{3.8}$$

The last step is to provide a well defined and coordinate invariant way of splitting the charge distribution Ω in decomposable pieces. It turns out that *the splitting in the minimal number of pieces is almost unique*. An antisymmetric matrix divides the tangent space into invariant subspaces of even dimensions. There is a couple of imaginary eigenvalues $i\lambda_k, -i\lambda_k$ associated with each dimension 2 invariant subspace. Each decomposable piece lives in one invariant subspace, and as long as the eigenvalues are different, the splitting is unique. With invariant subspaces of dimension bigger than 2 there can be ambiguities, but different splittings give the same DBI action. This concludes the argument.

We want to add some remarks on constraints posed by supersymmetry. For definiteness let us take Ω on the internal 6d manifold, which in our setup is a complex SU(3)-structure manifold. The internal geometry has an integrable complex structure $\mathcal{I}$ and a non-closed Kähler form $\mathcal{J}$ compatible with the metric: $\mathcal{J}_{ab} = g_{ac}\,\mathcal{I}_b{}^c$. We can find a vielbein basis that also diagonalizes the Kähler form:

$$\begin{aligned} \mathcal{J} &= \hat{e}^r \wedge \hat{e}^0 + \hat{e}^1 \wedge \hat{e}^2 + \hat{e}^3 \wedge \hat{e}^4 \\ g &= \sum\nolimits_a \hat{e}^a \otimes \hat{e}^a \,. \end{aligned} \tag{3.9}$$

This condition is invariant under the structure group SU(3) (without specifying the holomorphic 3-form, it is invariant under U(3)).

In our class of solutions, the supersymmetry equations force the charge distribution to be of type (1,1) with respect to the complex structure (see [42]).[18]

The decomposable pieces live in the invariant spaces of the antisymmetric matrix $(\Omega)_{ac}\, g^{cb}$ (the antisymmetry comes from Ω being of type (1,1) with respect to the complex structure compatible with the metric), while the moduli $\big|\Omega^{(k)}\big|$ are equal to the absolute values of the complex eigenvalues (which come in conjugated pairs) of the matrix:

$$\big(\Omega\big)_{ac}\, g^{cb} \qquad \longrightarrow \qquad \begin{cases} \big(\Omega^{(k)}\big)_{ac}\, g^{cb} & \text{on invariant spaces} \\ \big|\Omega^{(k)}\big| = |\lambda^{(k)}| & \text{complex eigenvalues} \end{cases}$$

Being practical, there is always a choice of vielbein basis which satisfies the diagonalizing conditions (3.9) and in which the charge distribution can be written as the sum of three (1,1) pieces:

$$\Omega = \lambda_1\, \hat{e}^r \wedge \hat{e}^0 + \lambda_2\, \hat{e}^1 \wedge \hat{e}^2 + \lambda_3\, \hat{e}^3 \wedge \hat{e}^4 \,. \tag{3.10}$$

[18] In complex notation, we have $e^{\phi}\bar{F}_1 = i\bar{\partial}\phi$ which implies $\bar{\partial}\bar{F}_1 = 0$. Being the charge distribution $\Omega = dF_1$, it must be (1,1).

We repeat that this splitting is unique as long as the three eigenvalues λ_k are different, while there are ambiguities for degenerate values but different choices give the same DBI action.

3.2 The BPS equations for any Sasaki-Einstein space

Let us now explain in detail the origin of the system of first-order differential equations (2.20). As already explained in section 2, the system (2.20) is a consequence of supersymmetry. Actually, it turns out that it can be derived in the more general situation that corresponds to having smeared D7-branes in a space of the type $AdS_5 \times M_5$, where M_5 is a five-dimensional Sasaki-Einstein (SE) manifold. Notice that the $T^{1,1}$ space considered up to now is a SE manifold. In general, a SE manifold can be represented as a one-dimensional bundle over a four-dimensional Kähler-Einstein (KE) space. Accordingly, we will write the M_5 metric as follows

$$ds^2_{\rm SE} = ds^2_{\rm KE} + (d\tau + A)^2 \ , \tag{3.11}$$

where $\partial/\partial\tau$ is a Killing vector and $ds^2_{\rm KE}$ stands for the metric of the KE space with Kähler form $J = dA\,/\,2$. In the case of the $T^{1,1}$ manifold the KE base is just $S^2 \times S^2$, where the S^2's are parametrized by the angles (θ_i, φ_i) and the fiber τ is parametrized by the angle ψ.

Our ansatz for ten-dimensional metric in Einstein frame will correspond to a deformation of the standard $AdS_5 \times M_5$. Apart from the ordinary warp factor $h(r)$, we will introduce some squashing between the one form dual to the Killing vector and the KE base, namely:

$$ds^2 \,=\, \Big[\,h(r)\Big]^{-\frac12} dx^2_{1,3} \,+\, \Big[\,h(r)\Big]^{\frac12} \Big[\,dr^2 \,+\, e^{2g(r)}\, ds^2_{\rm KE} \,+\, e^{2f(r)}\,\big(\,d\tau + A\big)^2\,\Big]. \tag{3.12}$$

Notice that, indeed, the ansatz (3.12) is of the same type as the one considered in eq. (1.10) for the deformation of $AdS_5 \times T^{1,1}$. In addition our background must have a Ramond-Ramond five form:

$$F_5 \,=\, K(r)\, dx^0 \wedge \cdots dx^4 \wedge dr \,+\, \text{Hodge dual}, \tag{3.13}$$

and a Ramond-Ramond one-form F_1 which violates Bianchi identity. Recall that this violation, which we want to be compatible with supersymmetry, is a consequence of having a smeared D7-brane source in our system. Our proposal for F_1 is the following:

$$F_1 \,=\, C\,(d\tau + A)\ , \tag{3.14}$$

where C is a constant which should be related to the number of flavors. Moreover, the violation of the Bianchi identity is the following:[19]

$$dF_1 \,=\, 2\,C\ J. \tag{3.15}$$

[19] We are considering that $J = \frac12 J_{ab} dx^a \wedge dx^b$ and that the Ricci tensor of the KE space satisfies $R_{ab} = 6\,g_{ab}$.

Notice that eq. (3.15) corresponds to taking $\Omega = 2CJ$ in our general expression (3.6). To proceed with this proposal we should try to solve the Killing spinor equations by imposing the appropriate projections. Notice that the ansatz is compatible with the Kähler structure of the KE base and this is usually related to supersymmetry.

Before going ahead, it may be useful for the interested reader to make contact with the explicit case studied in the previous section, namely the Klebanov-Witten model. In that case the KE base is

$$ds^2_{\rm KE} \,=\, \frac{1}{6}\sum_{i=1,2}(d\theta_i^2 \,+\, \sin^2\theta_i\, d\varphi_i^2) \tag{3.16}$$

whereas the one form dual to the Killing vector $\partial/\partial\tau$ is $d\tau = d\psi/3$ and the form A reads

$$A \,=\, \frac{1}{3}\Big(\cos\theta_1\, d\varphi_1 \,+\, \cos\theta_2\, d\varphi_2\Big)\,. \tag{3.17}$$

Moreover, the constant C was set to $\frac{3\,N_f}{4\pi}$ in that case.

Let us choose the following frame for the ten-dimensional metric:

$$\begin{aligned} \hat e^{x^\mu} &= \big[\,h(r)\,\big]^{-\frac14}\, dx^\mu & \hat e^{r} &= \big[\,h(r)\,\big]^{\frac14}\, dr\,, \\ \hat e^{0} &= \big[\,h(r)\,\big]^{\frac14}\, e^{f(r)}(d\tau + A) & \hat e^{a} &= \big[\,h(r)\,\big]^{\frac14}\, e^{g(r)}\, e^a, \end{aligned} \tag{3.18}$$

where $e^a \quad a = 1, \ldots, 4$ is the one-form basis for the KE space such that $ds^2_{\rm KE} \,=\, e^a\, e^a$. In the Klebanov-Witten model the basis taken in (2.14) corresponds to:

$$\begin{aligned} e^1 &= \sin\theta_1\, d\varphi_1\,, & e^2 &= d\theta_1\,, \\ e^3 &= \sin\theta_2\, d\varphi_2\,, & e^4 &= d\theta_2\,. \end{aligned} \tag{3.19}$$

Let us write the five-form $F_5 = \mathcal{F}_5 + {}^*\mathcal{F}_5$ of eq. (3.13) in frame components:

$$\begin{aligned} \mathcal{F}_5 &= K(r)\,\big[\,h(r)\,\big]^{\frac34}\, \hat e^{x^0}\wedge\cdots\wedge \hat e^{x^3}\wedge \hat e^{r}\,, \\ {}^*\mathcal{F}_5 &= -K(r)\,\big[\,h(r)\,\big]^{\frac34}\, \hat e^{0}\wedge\cdots\wedge \hat e^{4} \,=\, -Kh^2\, e^{4g+f}\,(d\tau + A)\wedge e^1\wedge\cdots\wedge e^4. \end{aligned} \tag{3.20}$$

The equation $dF_5 = 0$ immediately implies:

$$Kh^2 e^{4g+f} \,=\, {\rm constant} \,=\, \frac{(2\pi)^4 N_c}{Vol(M_5)}\,, \tag{3.21}$$

where the constant has been obtained by imposing the quantization condition (2.16) for a generic M_5. It will also be useful in what follows to write the one-form F_1 in frame components:

$$F_1 \,=\, C\, h^{-\frac14}\, e^{-f}\, \hat e^0. \tag{3.22}$$

Let us list the non-zero components of the spin connection:

$$\begin{aligned} \hat\omega^{x^\mu r} &= -\frac14\, h'\, h^{-\frac54}\; \hat e^{x^\mu}\,, && (\mu = 0, \cdots, 3)\,, \\ \hat\omega^{ar} &= \Big[\,\frac14\,\frac{h'}{h} \,+\, g'\,\Big]\, h^{-\frac14}\; \hat e^{a}\,, && (a = 1, \cdots, 4)\,, \\ \hat\omega^{0r} &= \Big[\,\frac14\,\frac{h'}{h} \,+\, f'\,\Big]\, h^{-\frac14}\; \hat e^{0}\,, && \\ \hat\omega^{0}{}_{a} &= e^{f-2g} h^{-\frac14}\, J_{ab}\, \hat e^{b}\,, && \\ \hat\omega^{ab} &= \omega^{ab} \,-\, e^{f-2g} h^{-\frac14}\, J^{ab} \hat e^{0}\,, && \end{aligned} \tag{3.23}$$

where ω^{ab} are components of the spin connection of the KE base.

Let us now study under which conditions our ansatz preserves some amount of supersymmetry. To address this point we must look at the supersymmetric variations of the dilatino (λ) and gravitino (ψ_M). These variations have been collected in appendix A, for both the Einstein and string frame. We have written them in eq. (2.17) for the particular case in which the three-forms of supergravity are zero. Recall that the variations written in (2.17) correspond to the Einstein frame and we have used a complex spinor notation.

It is quite obvious from the form of our ansatz for F_1 in (3.22) that the equation resulting from the dilatino variation is:

$$\big(\phi' + i\, e^{\phi}\, C\, e^{-f}\, \Gamma_{r0}\big)\, \epsilon \,=\, 0\;. \tag{3.24}$$

In eq. (3.24), and in what follows, the indices of the Γ-matrices refer to the vielbein components (3.18).

Let us move on to the more interesting case of the gravitino transformation. The space-time and the radial components of the equation do not depend on the structure of the internal space and always yield the following two equations:

$$\begin{aligned} h' + K\, h^2 \,&=\, 0\;, \\ \partial_r \epsilon - \frac{1}{8} K\, h\, \epsilon \,&=\, 0\;. \end{aligned} \tag{3.25}$$

To get eq. (3.25) we have imposed the D3-brane projection

$$\Gamma_{x^0 x^1 x^2 x^3}\, \epsilon \,=\, -i\, \epsilon\,, \tag{3.26}$$

and we have used the fact that the ten-dimensional spinor is chiral with chirality

$$\Gamma_{x^0 \dots x^3 r 01234}\, \epsilon \,=\, \epsilon\;. \tag{3.27}$$

It is a simple task to integrate the second differential equation in (3.25):

$$\epsilon \,=\, h^{-\frac{1}{8}} \hat{\epsilon}\;, \tag{3.28}$$

where $\hat{\epsilon}$ is a spinor which can only depend on the coordinates of the Sasaki-Einstein space.

In order to study the variation of the SE components of the gravitino it is useful to write the covariant derivative along the SE directions in terms of the covariant derivative in the KE space. The covariant derivative, written as a one-form for those components, $\hat{D} \equiv d + \frac{1}{4} \hat{\omega}_{IJ}\, \Gamma^{IJ}$, is given by

$$\begin{aligned} \hat{D} \,=\, D - \frac{1}{4} J_{ab}\, h^{-\frac{1}{4}}\, e^{f-2g}\, \Gamma^{ab}\, \hat{e}^0 - \frac{1}{2} J_{ab}\, h^{-\frac{1}{4}}\, e^{f-2g}\, \Gamma^{0b}\, \hat{e}^a \,+ \\ + \frac{1}{2} h^{-\frac{1}{4}} \big(\frac{1}{4} \frac{h'}{h} + g'\big)\, \Gamma^{ar}\, \hat{e}^a + \frac{1}{2} h^{-\frac{1}{4}} \big(\frac{1}{4} \frac{h'}{h} + f'\big)\, \Gamma^{0r}\, \hat{e}^0\;, \end{aligned} \tag{3.29}$$

where D is the covariant derivative in the internal KE space.

The equation for the SE components of the gravitino transformation is

$$\hat{D}_I\, \epsilon - \frac{1}{8} K\, h^{\frac{3}{4}}\, \Gamma_{rI}\, \epsilon + \frac{i}{4} e^{\phi}\, F^{(1)}_I\, \epsilon \,=\, 0. \tag{3.30}$$

This equation can be split into a part coming from the coordinates in the KE space and a part coming from the coordinate which parameterizes the Killing vector. For this purpose, it is convenient to represent the frame one-forms e^a and the fiber one-form A in a coordinate basis of the KE space

$$\begin{aligned} e^a &= E^a_m \, dy^m \,, \\ A &= A_m \, dy^m \,, \end{aligned} \tag{3.31}$$

with $y^m \quad m = 1, \ldots, 4$ a set of space coordinates in the KE space.

After a bit of algebra one can see that the equation obtained for the space coordinates y^m is simply

$$\begin{aligned} D_m \epsilon - \frac{1}{4} J_{ab} \, e^{2(f-g)} \, A_m \, \Gamma^{ab} \epsilon - \frac{1}{2} J_{ab} \, h^{-\frac{1}{4}} \, e^{f-2g} \, E^a_m \, \Gamma^{0b} \epsilon + \\ + \frac{1}{2} h^{-\frac{1}{4}} \left(\frac{1}{4} \frac{h'}{h} + g' \right) E^a_m \, \Gamma^{ar} \epsilon + \frac{1}{2} \left(\frac{1}{4} \frac{h'}{h} + f' \right) e^f \, A_m \, \Gamma^{0r} \epsilon - \\ - \frac{1}{8} K \, h^{\frac{3}{4}} \left(E^a_m \, \Gamma^{ra} + h^{\frac{1}{4}} \, e^f \, A_m \, \Gamma^{r0} \right) \epsilon + \frac{i}{4} e^{\phi} \, C \, A_m \, \epsilon = 0 \,, \end{aligned} \tag{3.32}$$

whereas the equation obtained for the fiber coordinate τ is given by

$$\begin{aligned} \frac{\partial \epsilon}{\partial \tau} - \frac{1}{4} J_{ab} \, e^{2(f-g)} \, \Gamma^{ab} \epsilon + \frac{1}{2} \left(\frac{1}{4} \frac{h'}{h} + f' \right) e^f \, \Gamma^{0r} \epsilon - \\ - \frac{1}{8} K \, h \, e^f \, \Gamma^{r0} \epsilon + \frac{i}{4} e^{\phi} \, C \, \epsilon = 0 \,. \end{aligned} \tag{3.33}$$

Let us now solve these equations for the spinor ϵ. First of all, let us consider the dilatino equation (3.24). Clearly, this equation implies that the spinor must be an eigenvector of the matrix Γ_{r0}. Accordingly, let us require that ϵ satisfies

$$\Gamma_{r0} \, \epsilon = - \, i \, \epsilon \,. \tag{3.34}$$

Moreover, a glance at eqs. (3.32) and (3.33) reveals that ϵ must also be an eigenvector of the matrix $J_{ab}\Gamma^{ab}$. Actually, by combining eqs. (3.26) , (3.27) and (3.34) one easily obtains that

$$\Gamma_{12}\epsilon = \Gamma_{34}\epsilon \,. \tag{3.35}$$

To simplify matters, let us assume that we have chosen the one-form basis e^a of the KE in such a way that the Kähler two-form J takes the canonical form:

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 \,. \tag{3.36}$$

In this basis, after using the condition (3.35), one trivially gets:

$$J_{ab}\Gamma^{ab} \, \epsilon = 4\Gamma_{12} \, \epsilon \,. \tag{3.37}$$

Thus, in order to diagonalize $J_{ab}\Gamma^{ab}$, let us impose the projection

$$\Gamma_{12} \, \epsilon = -i\epsilon \,, \tag{3.38}$$

which implies

$$\Gamma_{34}\,\epsilon \;=\; -i\epsilon\;, \qquad\qquad J_{ab}\Gamma^{ab}\,\epsilon \;=\; -4i\epsilon\;. \tag{3.39}$$

Let us now use the well-known fact that any KE space admits a covariantly constant spinor η satisfying:

$$D_m\,\eta \;=\; -\frac{3}{2}\,i\,A_m\,\eta\;, \tag{3.40}$$

from which one can get a Killing spinor of the five-dimensional SE space as:

$$\hat{\epsilon} \;=\; e^{-i\,\frac{3}{2}\tau}\,\eta\;. \tag{3.41}$$

Actually, in the KE frame basis we are using, η turns out to be a constant spinor which satisfies the conditions $\Gamma_{12}\,\eta \;=\; \Gamma_{34}\,\eta \;=\; i\eta$. Let us now insert the SE Killing spinor $\hat{\epsilon}$ of eq. (3.41) in our ansatz (3.28), i.e. we take the solution of our SUSY equations to be:

$$\epsilon \;=\; h^{-\frac{1}{8}}\,e^{-\frac{3}{2}\,i\tau}\,\eta\;. \tag{3.42}$$

By plugging (3.42) into eqs. (3.32) and (3.33), and using the projections imposed to ϵ and (3.40), one can easily see that eqs. (3.32) and (3.33) reduce to the following two differential equations:

$$\begin{aligned}
\frac{1}{4}\frac{h'}{h}\,+\,g'\,+\,\frac{1}{4}\,K\,h\,-\,e^{f-2g} \;&=\; 0\;,\\
\frac{1}{4}\frac{h'}{h}\,+\,f'\,+\,\frac{1}{4}\,K\,h\,+2\,e^{f-2g}\,-\,3\,e^{-f}\,+\,\frac{C}{2}e^{\phi-f} \;&=\; 0\;.
\end{aligned} \tag{3.43}$$

By combining all equations obtained so far in this section we arrive at a system of first-order BPS equations for the deformation of any space of the form $AdS_5\times M_5$:

$$\begin{aligned}
\phi'\,-\,C\,e^{\phi-f} \;&=\; 0\;,\\
h'\,+\,\frac{(2\pi)^4N_c}{Vol(M_5)}\,e^{-f-4g} \;&=\; 0\;,\\
g'\,-\,e^{f-2g} \;&=\; 0\;,\\
f'\,+2\,e^{f-2g}\,-\,3\,e^{-f}\,+\,\frac{C}{2}e^{\phi-f} \;&=\; 0\;.
\end{aligned} \tag{3.44}$$

Notice that, indeed, this system reduces to the one written in eq. (2.20) for the conifold, if we take into account that for this later case the constant C is $3N_f/(4\pi)$ and $Vol(T^{1,1}) = 16\pi^3/27$.

It is now a simple task to count the supersymmetries of the type (3.42) preserved by our background: it is just thirty-two divided by the number of independent algebraic projection imposed to the constant spinor η. As a set of independent projections one can take the ones written in eqs. (3.26), (3.34) and (3.38). It follows that our deformed background preserves four supersymmetries generated by Killing spinors of the type displayed in eq. (3.42).

3.3 The BPS and Einstein equations

In this section we will prove that the BPS system implies the fulfilment of the second-order Euler-Lagrange equations of motion for the combined gravity plus brane system. To begin with, let us consider the equation of motion of the dilaton, which can be written as:

$$\frac{1}{\sqrt{-G}}\partial_M\Big(G^{MN}\sqrt{-G}\,\partial_N\phi\Big) = e^{2\phi}F_1^2 - \frac{2\kappa_{10}^2}{\sqrt{-G}}\frac{\delta}{\delta\phi}S_{\rm DBI}\,, \tag{3.45}$$

where G_{MN} is the ten-dimensional metric. Using the DBI action (3.3) for the smeared D7-branes configuration, we find:

$$-\frac{2\kappa_{10}^2}{\sqrt{-G}}\frac{\delta}{\delta\phi}S_{\rm DBI} = e^{\phi}\sum\nolimits_i\big|\Omega^{(i)}\big|\,. \tag{3.46}$$

The charge density distribution is $\Omega = 2CJ$ (see eq. (3.15)). Recall that the Kähler form J of the KE base manifold has the canonical expression (3.36). It follows that Ω has two decomposable components given by:

$$\begin{aligned}\Omega^{(1)} &= 2C\,e^1\wedge e^2 = 2C\,h^{-\frac{1}{2}}\,e^{-2g}\,\hat e^1\wedge\hat e^2\,,\\ \Omega^{(2)} &= 2C\,e^3\wedge e^4 = 2C\,h^{-\frac{1}{2}}\,e^{-2g}\,\hat e^3\wedge\hat e^4\,,\end{aligned} \tag{3.47}$$

where the $\hat e^a$ one-forms have been defined in (3.18). Therefore, the moduli of the $\Omega^{(i)}$'s can be straightforwardly computed:

$$\big|\Omega^{(1)}\big| = \big|\Omega^{(2)}\big| = 2|\,C\,|\,h^{-\frac{1}{2}}\,e^{-2g}\,. \tag{3.48}$$

By using the explicit form of the metric, our ansatz for F_1 and the previous formulae (3.48) one can convert eq. (3.45) into the following:

$$\phi'' + (4g'+f')\,\phi' = C^2\,e^{2\phi-2f} + 4\,|C|\,e^{\phi-2g}\,. \tag{3.49}$$

It is now a simple exercise to verify that eq. (3.49) holds if the functions ϕ, g and f solve the first-order BPS system (3.44) and the constant C is non-negative. In what follows we shall assume that $C\geq 0$.

To check the Einstein equation we need to calculate the Ricci tensor. In flat coordinates the components of the Ricci tensor can be computed by using the spin connection. The expression of the curvature two-form in terms of the spin connection is

$$R_{\hat M\hat N} = d\hat\omega_{\hat M\hat N} + \hat\omega_{\hat M\hat P}\wedge\hat\omega^{\hat P}_{\ \hat N}\,, \tag{3.50}$$

with the curvature two-form defined as follows:

$$R^{\hat M}_{\ \hat N} = \frac{1}{2}\,R^{\hat M}_{\ \hat N\hat P\hat Q}\,e^{\hat P}\wedge e^{\hat Q}\,. \tag{3.51}$$

By using the values of the different components of the ten-dimensional spin connection written in (3.23) we can easily obtain the Riemann tensor and, by simple contraction of indices, we arrive at the following flat components of the Ricci tensor:

$$
\begin{aligned}
R_{x^ix^j} &= h^{-\frac{1}{2}}\eta_{x^ix^j}\left(\frac{1}{4}\frac{h''}{h}-\frac{1}{4}\left(\frac{h'}{h}\right)^2+\frac{1}{4}\frac{h'}{h}f'+\frac{h'}{h}g'\right),\\
R_{rr} &= h^{-\frac{1}{2}}\left(-\frac{1}{4}\frac{h''}{h}-\frac{1}{4}\left(\frac{h'}{h}\right)^2-\frac{1}{4}\frac{h'}{h}f'-\frac{h'}{h}g'-f''-(f')^2-4g''\right),\\
R_{00} &= h^{-\frac{1}{2}}\left(-\frac{1}{4}\frac{h''}{h}+\frac{1}{4}\left(\frac{h'}{h}\right)^2-\frac{1}{4}\frac{h'}{h}f'-\frac{h'}{h}g'-f''-(f')^2-4g'f'+4e^{2f-4g}\right),\\
R_{aa} &= h^{-\frac{1}{2}}\left(-\frac{1}{4}\frac{h''}{h}+\frac{1}{4}\left(\frac{h'}{h}\right)^2-\frac{1}{4}\frac{h'}{h}f'-\frac{h'}{h}g'-g''-\right.\\
&\qquad\qquad\left.-4(g')^2-g'f'-2e^{2f-4g}+6e^{-2g}\right),\\
R_{\hat{M}\hat{N}} &= 0, \qquad\qquad M\neq N.
\end{aligned}
\tag{3.52}
$$

From these values it is straightforward to find the expression of the scalar curvature, which is simply

$$
\begin{aligned}
R \,=\, -h^{-\frac{1}{2}}&\left(\frac{1}{2}\frac{h''}{h}+\frac{1}{2}\frac{h'}{h}f'+2\,\frac{h'}{h}g'\,+\,8\,g''+20\,(g')^2\,+\right.\\
&\left.+8\,g'\,f'+2\,f''+2\,(f')^2+4\,e^{2f-4g}-24\,e^{-2g}\right)\,.
\end{aligned}
\tag{3.53}
$$

Let us evaluate the different contributions to the right-hand side of Einstein's equations. The contributions from the five- and one-forms have been written in the first equation in (1.8) and is immediately computable from our ansatz of eqs. (3.13) and (3.14). On the other hand, the contribution of the DBI part of the action is just

$$
T_{MN} \,=\, -\frac{2\kappa_{10}^2}{\sqrt{-G}}\,\frac{\delta S_{\rm DBI}}{\delta G^{MN}}\,. \tag{3.54}
$$

By using our expression (3.3) of $S_{\rm DBI}$, with $\Omega = dF_1$, together with the definition (3.5), one easily arrives at the following expression of the stress-energy tensor of the D7-brane:

$$
T_{\hat{M}\hat{N}} \,=\, -\frac{e^{\phi}}{2}\,\Big[\eta_{\hat{M}\hat{N}}\,\sum_i\,|\,dF_1^{(i)}\,|\,-\,\sum_i\,\frac{1}{|\,dF_1^{(i)}\,|}\,(dF_1^{(i)})_{\hat{M}\hat{P}}\,(dF_1^{(i)})_{\hat{N}\hat{Q}}\,\eta^{\hat{P}\hat{Q}}\,\Big]\,, \tag{3.55}
$$

where we have used that $2\kappa_{10}^2 T_7 \,=\, 1$ and we have written the result in flat components. By using in (3.55) the values given in eqs. (3.47) and (3.48) of $dF_1^{(i)}$ and its modulus, we arrive at the simple result:

$$
\begin{aligned}
T_{x^ix^j} &= -2C\,h^{-\frac{1}{2}}\,e^{\phi-2g}\,\eta_{x^ix^j}\,,\\
T_{rr} &= T_{00} \,=\, -2C\,h^{-\frac{1}{2}}\,e^{\phi-2g}\,,\\
T_{ab} &= -C\,h^{-\frac{1}{2}}\,e^{\phi-2g}\,\delta_{ab}\,, \qquad\qquad (a,b=1,\cdots,4)\,,
\end{aligned}
\tag{3.56}
$$

where the indices refer to our vielbein basis (3.18). As a check of this result one can explicitly verify that the result of eq. (2.27) for the conifold, when written in flat indices, reduces to the simple expressions written in (3.56).

With all this information we can write, component by component, the set of second order differential equations for h, f and ϕ that are equivalent to the Einstein equations. One can then verify, after some calculation, that these equations are satisfied if ϕ and the functions of our ansatz solve the first-order system (3.44). Therefore, we have succeeded in proving that the background obtained from the supersymmetry analysis is a solution of the equations of motion of the supergravity plus Born-Infeld system. Notice that the SUSY analysis determines F_1, i.e. the RR charge distribution of the smeared D7-branes. What we have just proved is that eq. (3.55) gives the correct energy-momentum distribution associated to the charge distribution $\Omega = dF_1$ of the smeared flavor brane.

To finish this section let us write the DBI action in a different, and very suggestive, fashion. It turns out that, for our ansatz, the on-shell DBI action can be written as the integral of a ten-form and the corresponding expression is very similar to the one for the WZ term (eq. (3.1)). Actually, we show below that

$$S_{\rm DBI} \,=\, -T_7 \int_{\mathcal{M}_{10}} e^{\phi}\, dF_1 \wedge \Omega_8 \ , \tag{3.57}$$

where Ω_8 is an eight-form which, after performing the wedge product with the smearing two-form dF_1, gives rise to a volume form of the ten-dimensional space. Let us factorize in Ω_8 the factors coming from the Minkowski directions:

$$\Omega_8 \,=\, h^{-1}\, d^4x \wedge \Omega_4 \,, \tag{3.58}$$

where Ω_4 is a four-form in the internal space. Actually, one can check that Ω_4 can be written as:

$$\Omega_4 \,=\, \frac{1}{2}\, \mathcal{J} \wedge \mathcal{J}, \tag{3.59}$$

where $\mathcal{J}$ is the following two-form:

$$\mathcal{J} \,=\, h^{\frac{1}{2}}\, e^{2g}\, J \,+\, h^{\frac{1}{2}} e^{f}\, dr \wedge (d\tau \,+\, A)\,. \tag{3.60}$$

To verify this fact, let us recall that $dF_1 \,=\, 2CJ$ and thus

$$dF_1 \wedge \Omega_8 \,=\, C\, h^{-1}\, d^4x \,\wedge J \wedge \mathcal{J} \wedge \mathcal{J} \ . \tag{3.61}$$

Taking into account that $\frac{1}{2}\, J \wedge J$ is the volume form of the KE base of M_5, we readily get:

$$d^4x \,\wedge\, J \wedge \mathcal{J} \wedge \mathcal{J} \,=\, 4e^{-2g}\, h^{\frac{1}{2}}\, \sqrt{-G}\; d^{10}x \ , \tag{3.62}$$

from where one can easily prove that eq. (3.57) gives the same result as in equation (3.3) with $\Omega = dF_1$.

3.4 A superpotential and the BPS equations

It is interesting to obtain the system of first-order BPS equations (3.44) by using an alternative approach, namely by deriving them from a superpotential. Generically, let us consider a one-dimensional classical mechanics system in which η is the "time" variable and $\mathcal{A}(\eta)$, $\Phi^m(\eta)$ $(m = 1, 2\ldots)$ are the generalized coordinates. Let us assume that the Lagrangian of this system takes the form:

$$L \,=\, e^{\mathcal{A}}\Big[\kappa\,(\partial_\eta\mathcal{A})^2 \,-\, \frac{1}{2}G_{mn}(\Phi)\,\partial_\eta\Phi^m\,\partial_\eta\Phi^n \,-\, V(\Phi)\Big]\;, \tag{3.63}$$

where κ is a constant and $V(\Phi)$ is some potential, which we assume that is independent of the coordinate $\mathcal{A}$. If one can find a superpotential W such that:

$$V(\Phi) \,=\, \frac{1}{2}G^{mn}\,\frac{\partial W}{\partial\Phi^m}\,\frac{\partial W}{\partial\Phi^n} \,-\, \frac{1}{4\kappa}\,W^2\;, \tag{3.64}$$

then the equations of motion are automatically satisfied by the solutions of the first order system:

$$\frac{d\,\mathcal{A}}{d\eta} \,=\, -\frac{1}{2\kappa}W\;, \qquad \frac{d\,\Phi^m}{d\eta} \,=\, G^{mn}\,\frac{\partial W}{\partial\Phi^n}\;. \tag{3.65}$$

Let us now show how we can recover our system (3.44) from this formalism. The first step is to look for an effective Lagrangian for the dilaton and the functions of our ansatz whose equations of motion are the same as those obtained from the Einstein and dilaton equations of Type IIB supergravity. One can see that this lagrangian is:

$$L_{\rm eff} = h^{\frac{1}{2}}e^{4g+f}\Big[R-\frac{1}{2}h^{-\frac{1}{2}}(\phi')^2-\frac{Q^2}{2}h^{-\frac{5}{2}}e^{-8g-2f}-\frac{C^2}{2}h^{-\frac{1}{2}}\,e^{2\phi-2f}-4Ch^{-\frac{1}{2}}\,e^{\phi-2g}\Big]\;, \tag{3.66}$$

where R is the scalar curvature as written in (3.53) and Q is the constant

$$Q \,\equiv\, \frac{(2\pi)^4N_c}{Vol(M_5)}\;. \tag{3.67}$$

The Ricci scalar (3.53) contains second derivatives. Up to total derivatives $L_{\rm eff}$ takes the form:

$$\begin{aligned} L_{\rm eff} \,=\, e^{4g+f}\Bigg[&-\frac{1}{2}\Big(\frac{h'}{h}\Big)^2 \,+\, 12\,(g')^2 \,+\, 8\,g'\,f' \,-\, 4\,e^{2f-4g} \,+\, 24\,e^{-2g} \,-\, \frac{1}{2}(\phi')^2 \,- \\ &-\frac{Q^2}{2}\,h^{-2}\,e^{-8g-2f} \,-\, \frac{C^2}{2}\,e^{2(\phi-f)} \,-\, 4\,C\,e^{\phi-2g}\Bigg]\;. \end{aligned} \tag{3.68}$$

We want to pass from the lagrangian (3.68) to that in eq. (3.63). With that purpose in mind let us perform the following redefinition of fields:

$$e^{\frac{3}{4}\mathcal{A}} \,=\, h^{\frac{1}{2}}e^{4g+f}\;, \qquad e^{2\tilde g} \,=\, h^{\frac{1}{2}}e^{2g}, \qquad e^{2\tilde f} \,=\, h^{\frac{1}{2}}e^{2f}. \tag{3.69}$$

In addition, we need to do the following change of the radial variable [20]

$$\frac{dr}{d\eta} \,=\, e^{\frac{\mathcal{A}}{4}-\frac{8}{3}\tilde g-\frac{2}{3}\tilde f}\;. \tag{3.70}$$

[20]The change of the Lagrangian under that change of radial variable is $\hat L_{\rm eff} = \frac{dr}{d\eta}\,L_{\rm eff}$.

Once we have done the previous redefinitions, the Lagrangian we obtain is:

$$\hat{L}_{eff} = e^{\mathcal{A}}\Bigg[\frac{3}{4}\,(\dot{\mathcal{A}})^2 - \frac{28}{3}\,(\dot{\tilde{g}})^2 - \frac{4}{3}(\dot{\tilde{f}})^2 - \frac{8}{3}\,\dot{\tilde{g}}\,\dot{\tilde{f}} - \frac{1}{2}(\dot{\phi})^2 - V(\tilde{g},\tilde{f},\phi)\Bigg]\,, \tag{3.71}$$

where the dot means derivative with respect to η and $V(\tilde{g},\tilde{f},\phi)$ is the following potential:

$$V(\tilde{g},\tilde{f},\phi) = e^{-\frac{2}{3}(4\tilde{g}+\tilde{f})}\left(4e^{2\tilde{f}-4\tilde{g}} - 24e^{-2\tilde{g}} + \frac{Q^2}{2}e^{-2(4\tilde{g}+\tilde{f})} + \frac{C^2}{2}e^{2(\phi-\tilde{f})} + 4Ce^{\phi-2\tilde{g}}\right)\,. \tag{3.72}$$

The above lagrangian has the desired form (see eq. (3.63)) and we can identify the constant κ and the elements of the kinetic matrix G_{mn} as:

$$\kappa = \frac{3}{4}\,,\quad G_{\tilde{g}\tilde{g}} = \frac{56}{3}\,,\quad G_{\tilde{f}\tilde{f}} = \frac{8}{3}\,,\quad G_{\tilde{g}\tilde{f}} = \frac{8}{3}\,,\quad G_{\phi\phi} = 1\,. \tag{3.73}$$

One can now check that, given the above expression of the potential, the following superpotential

$$W = e^{-\frac{1}{3}(4\tilde{g}+\tilde{f})}\Big[Q\,e^{-4\tilde{g}-\tilde{f}} - 4\,e^{\tilde{f}-2\tilde{g}} - 6e^{-\tilde{f}} + Ce^{\phi-\tilde{f}}\Big] \tag{3.74}$$

satisfies eq. (3.64) for the values of κ and G_{mn} written in eq. (3.73). It is now immediate to write the first-order differential equations that stem from this superpotential. Explicitly we obtain:

$$\begin{aligned}
\dot{\mathcal{A}} &= -\frac{2}{3}\,W\,,\\
\dot{\tilde{g}} &= \frac{1}{4}e^{-\frac{1}{3}(4\tilde{g}+\tilde{f})}\Big[-Qe^{-4\tilde{g}-\tilde{f}} + 4\,e^{\tilde{f}-2\tilde{g}}\Big]\,,\\
\dot{\tilde{f}} &= \frac{1}{4}\,e^{-\frac{1}{3}(4\tilde{g}+\tilde{f})}\Big[-Qe^{-4\tilde{g}-\tilde{f}} - 8\,e^{\tilde{f}-2\tilde{g}} + 12\,e^{-\tilde{f}} - 2\,C\,e^{\phi-\tilde{f}}\Big]\,,\\
\dot{\phi} &= C\,e^{\phi-\frac{4}{3}(\tilde{g}+\tilde{f})}\,.
\end{aligned} \tag{3.75}$$

In order to verify that this system is equivalent to the one obtained from supersymmetry, let us write down explicitly these equations in terms of the old radial variable (see eq. (3.70)) and fields (see eqs. (3.69)). One gets:

$$\begin{aligned}
\frac{h'}{h} + 8\,g' + 2\,f' &= -Q\,h^{-1}\,e^{-4g-f} + 4\,e^{f-2g} + 6e^{-f} - Ce^{\phi-f}\,,\\
\frac{1}{4}\frac{h'}{h} + g' &= e^{f-2g} - \frac{1}{4}Q\,h^{-1}\,e^{-4g-f}\,,\\
\frac{1}{4}\frac{h'}{h} + f' &= 3\,e^{-f} - 2\,e^{f-2g} - \frac{1}{4}Q\,h^{-1}\,e^{-4g-f} - \frac{1}{2}C\,e^{\phi-f}\,,\\
\phi' &= C\,e^{\phi-f}\,,
\end{aligned} \tag{3.76}$$

which are nothing else than a combination of the system of BPS equations written in (3.44).

Let us now use the previous results to study the 5d effective action resulting from the compactification along M_5 of our solution. The fields in this effective action are the functions $\tilde{f}$ and $\tilde{g}$, which parameterize the deformations along the fiber and the KE base of

M_5 respectively, and the dilaton. Actually, in terms of the new radial variable η introduced in (3.70), the ten-dimensional metric can be written as:

$$ds^2 \,=\, e^{-\frac{2}{3}\,(\tilde f+4\tilde g)}\,\Big[e^{\frac{\mathcal{A}}{2}}\,dx^\mu dx_\mu \,+\, d\eta^2\Big] \,+\, e^{2\tilde g}\,ds^2_{\rm KE} \,+\, e^{2\tilde f}\,(d\tau + A)^2\,. \tag{3.77}$$

The corresponding analysis for the unflavored theory was performed in [8, 35]. For simplicity, let us work in units in which the AdS_5 radius L is one. Notice that the quantity Q defined in (3.67) is just $Q = 4L^4$. Thus, in these units $Q = 4$. To make contact with the analysis of refs. [8, 35], let us introduce new fields q and p which, in terms of $\tilde f$ and $\tilde g$ are defined as follows:[21]

$$q \,=\, \frac{2}{15}\,(\,\tilde f \,+\, 4\,\tilde g\,)\,, \qquad\qquad p \,=\, -\,\frac{1}{5}\,(\,\tilde f \,-\, \tilde g\,)\,. \tag{3.78}$$

In terms of these new fields, the potential (3.72) turns out to be

$$V(p,q,\phi) \,=\, 4\,e^{-8q-12p} \,-\, 24\,e^{-8q-2p} \,+\, \frac{C^2}{2}\,e^{2\phi-8q+8p} \,+\, 8\,e^{-20\hat q} \,+\, 4\,C\,e^{\phi-8q+8p}\,, \tag{3.79}$$

and the effective lagrangian (3.71) can be written as:

$$L_{\rm eff} \,=\, \sqrt{-g_5}\,\Big[\,R_5 \,-\, \frac{1}{2}\,\dot\phi^2 \,-\, 20\,\dot p^2 \,-\, 30\,\dot q^2 \,-\, V\,\Big]\,, \tag{3.80}$$

where $g_5 \,=\, -e^{2\mathcal{A}}$ is the determinant of the five-dimensional metric $ds^2_5 \,=\, e^{\frac{\mathcal{A}}{2}}\,dx^\mu dx_\mu + d\eta^2$ and $R_5 \,=\, -[2\,\ddot{\mathcal{A}} \,+\, \frac{5}{4}\,\dot{\mathcal{A}}^2\Big]$ is its Ricci scalar. One can check that the minimum of the potential (3.79) occurs only at $p \,=\, q \,=\, e^\phi \,=\, 0$, which corresponds to the conformal $AdS_5 \times M_5$ geometry. Moreover, by expanding V around this minimum at second order we find out that the fields p and q defined in (3.78) diagonalize the quadratic potential. The corresponding masses are $m_p^2 = 12$ and $m_q^2 = 32$. By using these values in the mass-dimension relation (2.86), we get:

$$\begin{aligned} m_p^2 = 12 \;&\Longrightarrow\; \Delta_p \,=\, 6\,,\\ m_q^2 = 32 \;&\Longrightarrow\; \Delta_p \,=\, 8\,. \end{aligned} \tag{3.81}$$

These scalar modes p and q are dual to the dimension 6 and 8 operators discussed in section 2.

3.5 General deformation of the Klebanov-Witten background

In this section we will explore the possibility of having a more general flavor deformation of the $AdS_5 \times T^{1,1}$ background. Notice that, as $T^{1,1}$ is a U(1) bundle over $S^2 \times S^2$, there exists the possibility of squashing with different functions each of the two S^2's of the KE base. In the unflavored case this is precisely the type of deformation that occurs when the singular conifold is substituted by its small resolution. For this reason, it is worth to

[21]The function p is called f in refs. [8, 35].

consider this type of metric also in our flavored background. To be precise, let us adopt the following ansatz for the metric, five-form and one-form:

$$\begin{aligned}
ds^2 &= h^{-1/2} dx_{1,3}^2 + h^{1/2}\Big\{ dr^2 + \frac{1}{6}\sum_{i=1,2} e^{2g_i}(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2) + \\
&\qquad\qquad + \frac{e^{2f}}{9}\Big(d\psi + \sum_{i=1,2}\cos\theta_i\, d\varphi_i\Big)^2 \Big\} \, , \\
F_5 &= (1+*)\, d^4x \wedge K\, dr \, , \\
F_1 &= \frac{C}{3}(d\psi + \cos\theta_2\, d\varphi_2 + \cos\theta_1\, d\varphi_1) \, ,
\end{aligned} \tag{3.82}$$

where $C = 3N_f/4\pi$, all functions depend on r and $g_1(r)$ and $g_2(r)$ are, in general, different (if $g_1 = g_2 = g$ we recover our ansatz (1.10)). The equation $dF_5 = 0$ immediately implies:

$$K h^2 e^{2g_1+2g_2+f} = 27\pi N_c \equiv Q \, , \tag{3.83}$$

which allows to eliminate the function K in favor of the other functions of the ansatz. By following the same steps as in the $g_1 = g_2$ case and requiring that the background preserve four supersymmetries, we get a system of first-order BPS equations for this kind of deformation, namely:

$$\begin{aligned}
\partial_r\phi &= C\, e^{\phi-f} \, , \\
h' &= -Q\, e^{-f-2g_1-2g_2} \, , \\
g_i' &= e^{f-2g_i} \, , \qquad\qquad (i=1,2) \, , \\
f' &= 3\, e^{-f} - e^{f-2g_1} - e^{f-2g_2} - \frac{C}{2}\, e^{\phi-f} \, .
\end{aligned} \tag{3.84}$$

Notice that, as it should, the system (3.84) reduces to eq. (3.44) when $g_1 = g_2$.

It is not difficult to integrate this system of differential equations by following the same method that was employed for the $g_1 = g_2$ case. First of all, we change the radial coordinate:

$$dr = e^f\, d\rho \, , \tag{3.85}$$

what allows us to get a new system:

$$\begin{aligned}
\dot\phi &= C\, e^{\phi} \, , \\
\dot h &= -Q\, e^{-2g_1-2g_2} \, , \\
\dot g_i &= e^{2f-2g_i} \, , \qquad\qquad (i=1,2) \, , \\
\dot f &= 3 - e^{2f-2g_1} - e^{2f-2g_2} - \frac{C}{2}\, e^{\phi} \, ,
\end{aligned} \tag{3.86}$$

where now the derivatives are taken with respect to the new variable ρ.

The equation for the dilaton in (3.86) can be integrated immediately, with the result:

$$e^{\phi} = -\frac{1}{C}\,\frac{1}{\rho} \, , \qquad\qquad (\rho < 0) \, , \tag{3.87}$$

where we have absorbed an integration constant in a shift of the radial coordinate. Moreover, by combining the equations for g_1 and g_2 one easily realizes that the combination $e^{2g_1} - e^{2g_2}$ is constant. Let us write:

$$e^{2g_1} = e^{2g_2} + a^2 . \tag{3.88}$$

On the other hand, by using the solution for $\phi(r)$ just found and the equations for the g_i's in (3.86), the first-order equation for f can be rewritten as:

$$\dot{f} = 3 - \dot{g}_1 - \dot{g}_2 + \frac{1}{2\rho} , \tag{3.89}$$

which can be integrated immediately, to give:

$$e^{2f+2g_1+2g_2} = -c\rho e^{6\rho} , \tag{3.90}$$

with c being an integration constant. This constant can be absorbed by performing a suitable redefinition. In order to make contact with the case in which $g_1 = g_2$ let us take $c = 6$. Then, by combining (3.90) with the equation of g_2, we get

$$e^{4g_2+2g_1}\,\dot{g}_2 = e^{2g_1+2g_2+2f} = -6\rho e^{6\rho} , \tag{3.91}$$

which, after using the relation (3.88), can be integrated with the result

$$e^{6g_2} + \frac{3}{2}\,a^2\,e^{4g_2} = (1-6\rho)\,e^{6\rho} + c_1 . \tag{3.92}$$

Notice that, indeed, for $a = 0$ this equation reduces to the $g_1 = g_2$ solution (see eq. (2.43)). Moreover, by combining eqs. (3.88) and (3.90) the expression of f can be straightforwardly written in terms of g_2, as follows:

$$e^{2f} = -\frac{6\rho e^{6\rho}}{e^{4g_2} + a^2\,e^{2g_2}} . \tag{3.93}$$

It is also easy to get the expression of the warp factor h:

$$h(\rho) = -Q \int \frac{d\rho}{e^{4g_2} + a^2\,e^{2g_2}} + c_2 . \tag{3.94}$$

Thus, the full solution is determined in terms of e^{2g_2} which, in turn, can be obtained from (3.92) by solving a cubic algebraic equation. In order to write the explicit value of e^{2g_2}, let us define the function:

$$\xi(\rho) \equiv (1-6\rho)\,e^{6\rho} + c_1 . \tag{3.95}$$

Then, one has:

$$e^{2g_2} = \frac{1}{2}\left[-a^2 + \frac{a^4}{\big[\zeta(\rho)\big]^{\frac{1}{3}}} + \big[\zeta(\rho)\big]^{\frac{1}{3}} \right] , \tag{3.96}$$

where the function $\zeta(\rho)$ is defined in terms of $\xi(\rho)$ as:

$$\zeta(\rho) \equiv 4\,\xi(\rho) - a^6 + 4\sqrt{\xi(\rho)^2 - \frac{a^6}{2}\,\xi(\rho)}\,. \tag{3.97}$$

In expanding these functions in series near the UV ($\rho \to 0$) one gets a similar behavior to the one discussed in section 2.3. Very interestingly, in the IR of the field theory, that is when $\rho \to -\infty$, we get a behavior that is "softened" respect to what we found in section 2.3. This is not unexpected, given the deformation parameter a. Nevertheless, the solutions are still singular. Indeed, the dilaton was not affected by the deformation a.

3.6 Massive flavors

In the ansatz we have been using up to now we have assumed that the density of RR charge of the D7-branes is independent of the holographic coordinate. This is, of course, what is expected for a flavor brane configuration which corresponds to massless quarks. On the contrary, in the massive quark case, a supersymmetric D7-brane has a non-trivial profile in the radial direction [20] and, in particular ends at some non-zero value of the radial coordinate. These massive embeddings have free parameters which could be used to smear the D7-branes. It is natural to think that the corresponding charge and mass distribution of the smeared flavor branes will depend on the radial coordinate in a non-trivial way.

It turns out that there is a simple modification of our ansatz for F_1 which gives rise to a charge and mass distribution with the characteristics required to represent smeared flavor branes with massive quarks. Indeed, let us simply substitute in (3.44) the constant C by a function $C(r)$. In this case:

$$\begin{aligned} F_1 &= C(r)\,(d\tau + A)\,, \\ dF_1 &= 2\,C(r)\,J + C'(r)dr \wedge (d\tau + A)\,. \end{aligned} \tag{3.98}$$

Notice that the SUSY analysis of section 3.2 remains unchanged since only F_1, and not its derivative, appears in the supersymmetric variations of the dilatino and gravitino. The final result is just the same system (3.44) of first-order BPS equations, where now one has to understand that C is a prescribed function of r, which encodes the non-trivial profile of the D7-brane. Notice that $C(r)$ determines the running of the dilaton which, in turn, affects the other functions of the ansatz.

A natural question to address here is whether or not the solutions of the modified BPS system solve the equations of motion of the supergravity plus branes system. In order to check this fact, let us write the DBI term of the action, following our prescription (3.3). Notice that, in the present case, $\Omega = dF_1$ is the sum of three decomposable pieces:

$$dF_1 = dF_1^{(1)} + dF_1^{(2)} + dF_1^{(3)}\,, \tag{3.99}$$

where $dF_1^{(1)}$ and $dF_1^{(2)}$ are just the same as in eq. (3.47), while $dF_1^{(3)}$ is given by:

$$dF_1^{(3)} = C'(r)\,dr \wedge (d\tau + A) = h^{-\frac{1}{2}}\,e^{-f}\,C'(r)\;e^r \wedge e^0\,. \tag{3.100}$$

The modulus of this new piece of dF_1 can be straightforwardly computed, namely:

$$|\,dF_1^{(3)}\,| \;=\; h^{-\frac{1}{2}}\,e^{-f}\,|\,C'(r)\,|\;. \qquad (3.101)$$

By using this result, together with the one in (3.48), one readily gets the expression of the DBI terms of the action of the smeared D7-branes:

$$S_{\rm DBI} \;=\; -\,T_7\int_{\mathcal{M}_{10}} h^{-\frac{1}{2}}\,e^{\phi}\,\Big(4\,|\,C(r)\,|\,e^{-2g} \,+\, |\,C'(r)\,|\,e^{-f}\Big)\,\sqrt{-G}\;d^{10}x\,. \qquad (3.102)$$

From this action it is immediate to find the equation of motion of the dilaton, i.e.:

$$\phi'' \,+\, (4g'+f')\,\phi' \;=\; C^2\,e^{2\phi-2f} \,+\, 4\,|C|\;e^{\phi-2g} \,+\, e^{\phi-f}\,|\,C'\,|\;. \qquad (3.103)$$

It can be verified that the first-oder BPS equations (3.44) imply the fulfilment of eq. (3.103), provided the functions $C(r)$ and $C'(r)$ are non-negative. Notice that now, when computing the second derivative of ϕ from the BPS system (3.44) with $C = C(r)$, a new term containing $C'(r)$ is generated. It is easy to verify that this new term matches precisely the last term on the right-hand side of (3.103).

It remains to verify the fulfilment of Einstein's equation. The stress-energy tensor of the brane can be computed from eq. (3.55), where now the extra decomposable piece of dF_1 must be taken into account. The result one arrives at, in the vielbein basis (3.18), is a direct generalization of (3.56):

$$\begin{aligned}
T_{x^ix^j} \;&=\; -\,e^{\phi}\,h^{-\frac{1}{2}}\Big[2\,|\,C(r)\,|\,e^{-2g} \,+\, \frac{1}{2}\,|\,C'(r)\,|\,e^{-f}\Big]\,\eta_{x^ix^j}\;, \qquad (i,j=0,\ldots,3)\;,\\
T_{ab} \;&=\; -\,e^{\phi}\,h^{-\frac{1}{2}}\Big[\,|\,C(r)\,|\,e^{-2g} \,+\, \frac{1}{2}\,|\,C'(r)\,|\,e^{-f}\Big]\,\delta_{ab}\;, \qquad (a,b=1,\ldots,4)\;,\\
T_{rr} \;=\; T_{00} \;&=\; -\,2\,|\,C(r)\,|\;h^{-\frac{1}{2}}\,e^{\phi-2g}\;.
\end{aligned} \qquad (3.104)$$

As happened for the equation of motion of the dilaton, one can verify that the extra pieces on the right-hand side of (3.104) match precisely those generated by the second derivatives appearing in the expression (3.52) of the Ricci tensor if $C(r)$ and $C'(r)$ are non-negative. As a consequence, the first-order equations (3.44) with a function $C(r)$ also imply the equations of motion for the ten-dimensional metric g_{MN}. It is also interesting to point out that, if $C(r)$ and $C'(r)$ are non-negative, $S_{\rm DBI}$ can also be written in the form (3.57), where Ω_8 is exactly the same eight-form as in eqs. (3.58) and (3.59).

Notice that, if the function $C(r) = 3N_f(r)/4\pi$ has a Heaviside-like shape "starting" at some finite value of the radial coordinate, then our BPS equations and solutions will be the ones given in section 2.3 for values of the radial coordinate bigger than the "mass of the flavor". However, below that radial value the solution will be the one of Klebanov-Witten (or deformations of it, see appendix B), with a non-running dilaton. Aside from decoupling in the field theory, this is clearly indicating that the addition of massive flavors "resolves" the singularity. Physically this behavior is expected and makes these massive flavor more interesting.

4. Summary, future prospects and further discussion

In this paper we followed the method of [17] to construct a dual to the field theory defined by the Klebanov and Witten after N_f flavors of quarks and antiquarks have been added to both gauge groups. In section 2 of this work, we wrote BPS equations describing the dynamics of this system and found solutions to this first order system, that of course solves also all the second order equations of motion. We analyzed the solutions to the BPS system and learnt that, even when singular, the character of the singularity permits to get field theory conclusions from the supergravity perspective.

We proposed a formulation for the dual field theory to these solutions, constructing a precise 4-dimensional superpotential. We studied these solutions making many matchings with field theory expectations that included the R-symmetry breaking and Wilsonian beta function. Also, using the well known (supergravity) superpotential approach, we learnt that our field theory, aside from being deformed by a marginal (then turned irrelevant) operator, modifies its dynamics by giving VEV to operators of dimension six and eight. We explained how to change relations between couplings and θ-angles in the theory, from the perspective of our solutions. We believe that these many checks should encourage other physicists to study this background more closely.

In section 3 of this paper, we presented a careful account of the many technical details regarding the derivation of the results in section 2 summarized above. But most interestingly, section 3 is not only about technical details. Indeed, using the logic and intuitions developed in section 2, we generalized the approach described there for *any* five dimensional manifold that can be written as a Sasaki-Einstein space (a one-dimensional fibration over a Kähler-Einstein space). It is surprising that the same structure of BPS eqs and ten-dimensional superpotential repeats for all the manifolds described above. This clearly points to some "universality" of the behavior of 4-dimensional $\mathcal{N} = 1$ SCFT's with flavors.

We have added some brief comments about what happens when we take the number of flavors $N_f = 0$ in our BPS eqs (see appendix B). It is interesting to recover some solutions studied in the past from this perspective since it puts into context previous analysis. Again, the careful study of this "unflavored" solutions might be of interest to many physicists. We shortly commented on the possibility of adding to the dynamics of the 4-d field theory fundamentals with mass, presenting a general context to do this. We will exploit this procedure in the future to get a better understanding of our singular backgrounds, make contact with field theory results and study many other interesting problems.

All the results described above not only motivate a more detailed analysis of this approach from a field theoretical viewpoint, but also emphasize the need for a deeper geometrical study, that clearly will reveal interesting underlying structure.

4.1 Future directions

Many things can be done following the results of this paper. It is natural to extend the treatment to the case of the Klebanov-Tseytlin and Klebanov-Strassler solutions. The result is likely to be interesting, since the fundamentals and the KT cascade "push in

JHEP02(2007)090

different directions" in the RG flow. One might find a fine-tuned situation in which the IR dynamics is different from the one in the Klebanov-Strassler model.

Other things that immediately come to mind are to study the dynamics of moving strings in this backgrounds, details related to dibaryons, flavor symmetry breaking, etc. Even when technically involved, it should be nice to understand the backreaction of probes where the worldvolume fields have been turned on, since some interesting problems may be addressed.

The formalism developed to deal with configurations of IIB dual to massive fundamentals seems useful in different contexts. Needless to say, the approach adopted here is immediately generalizable to the case of type IIA backgrounds. Duals to $\mathcal{N}=1$ field theories have been constructed and it seems natural to apply our methods in those cases.

Finding black hole solutions in our geometries is not an elementary task; but it should not be very difficult. The interest of this problem resides in the fact that this will produce a "well-defined" black hole background where to study, among other things, plasmas that include the dynamics of color and flavor at strong coupling. This is a very well defined problem that we believe of much interest.

On the field theory side, it should be interesting to understand in more detail how the smearing procedure affects the superpotential. We gave a possible answer and detailed study can uncover interesting subtleties. Here again, similar ideas can be extended to other situations in type IIA and type IIB. Getting a better handle on the field theory interpretation of our "generalized" approach of Part II seems also interesting. Indeed, understanding in detail what is the "universality" that produces the same dynamics for a large class of $\mathcal{N}=1$ SCFT's with flavor would be nice.

4.2 Further discussion

Let us finish this paper with some discussions that might be of interest for the reader. The first point we want to address is what could be the application of these results to Physics. Indeed, it is not easy to find an interesting physical system displaying a Landau pole (without a UV completion, like QED has, for example). Of course, as explained above, this paper is a first step in a more detailed study of a cascading field theory with flavors, that with no doubt has applications in Physics. Nevertheless, one can find some interesting problems already at this stage.

As described above, finding a black hole in our geometry, might be a good simulation of the Physics of a strongly coupled quark-gluon plasma. Even more, since we would be only interested in effects in the hydrodynamics regime, using the IR of this black hole solutions should be enough to learn about Physics at RHIC, for example. One can also think that our paper starts the study of the different phases of this generalized $\mathcal{N}=1$ SQCD obtained from Klebanov-Witten-like models.

Let us change the subject of the discussion and go back to our procedure, that was well explained in the introduction of this paper. The reader may remember the difference between a weakly gauged symmetry and a global symmetry, let us now connect this to supergravity. One important distinction between the approaches for finding string duals to field theories with fundamentals is that in the approach where the solution consists only of supergravity fields, the field theory will have this global symmetry weakly gauged. On the contrary, in our case backreacting with the Born-Infeld action, the symmetry will be global. We can see this clearly in the fact that the BI action has the freedom to add worldvolume gauge fields (and scalars), hence introducing a gauged symmetry in the bulk, dual to a global symmetry in the boundary. In a (complicated) reduction of our Type IIB plus Born-Infeld action to five dimensions, we would see some SU(N_f) gauge fields (as many as branches of flavor branes we added) that would enter in the holographic formulas to compute field theory correlators.

It is interesting to notice that depending on the physical situation we want to work with, we should choose the approach used here or the complementary one of finding a solution purely in supergravity. Indeed, for situations where we do not want to take into account the "flavor degrees of freedom" of the extra branes, but what we want is to introduce some operator in the dual field theory (like a giant graviton, a domain wall or a Wilson line) we should work within the purely supergravity approach [43]. Indeed, if we are thinking about the presence of an operator (say in $N = 4$ SYM), there should be no "flavor degrees of freedom" in the solutions.

Finally, we would like to comment on the smearing procedure. One way in which we can think about it is to realize that usually (unless they are D9 branes) the "localized" flavor branes will break part of the isometries of the original background dual to the unflavored field theory. The "smeared" flavor branes on the other hand reinstate these isometries (global symmetries of the field theory dual). In some sense the flavor branes are 'deconstructing' these dimensions (or these global groups) for the field theory of interest. In the case in which we have a finite number of flavors, these manifolds become fuzzy, while for $N_f \to \infty$, we recover the full invariance.

Acknowledgments

Some physicists helped us with discussions, conversations and correspondence, to understand and present the results of this paper in a better way. We would like to thank Daniel Areán, Adi Armoni, Sergio Benvenuti, Gaetano Bertoldi, Matteo Bertolini, Nadav Drukker, Jose D. Edelstein, Davide Forcella, Tim Hollowood, Carlos Hoyos, Emiliano Imeroni, Harald Ita, Prem Kumar, Javier Mas, Paolo Merlatti, Asad Naqvi, Angel Paredes, Rodolfo Russo, Kostas Skenderis and Alberto Zaffaroni for their very valuable input. The work of FC and AVR was supported in part by MCyT and FEDER under grant FPA2005-00188, by Xunta de Galicia (Conselleria de Educacion and grant PGIDIT06PXIB206185PR) and by the EC Commission under grant MRTN-CT-2004-005104.

A. SUSY transformations in string and Einstein frame

The supersymmetry transformations of Type IIB supergravity were found long ago in ref. [44]. Here we will follow the conventions of the appendix A of [45], where they are written in string frame. Let us recall them:

$$
\begin{aligned}
\delta_\epsilon \lambda^{(s)} &= \frac{1}{2}\Big(\Gamma^{(s)\,M}\partial_M\phi + \frac{1}{2}\frac{1}{3!}H_{MNP}\,\Gamma^{(s)\,MNP}\,\sigma_3\Big)\epsilon^{(s)} - \frac{1}{2}e^{\phi}\Big(F^{(1)}_M\,\Gamma^{(s)\,M}\,(i\sigma_2) + \\
&+ \frac{1}{2}\frac{1}{3!}F^{(3)}_{MNP}\,\Gamma^{(s)\,MNP}\,\sigma_1\Big)\epsilon^{(s)}, \\
\delta_\epsilon \psi^{(s)}_M &= \nabla^{(s)}_M\epsilon^{(s)} + \frac{1}{4}\frac{1}{2!}H_{MNP}\,\Gamma^{(s)\,NP}\,\sigma_3\epsilon^{(s)} + \frac{1}{8}e^{\phi}\Big(F^{(1)}_N\,\Gamma^{(s)\,N}\,(i\sigma_2) + \\
&+ \frac{1}{3!}F^{(3)}_{NPQ}\,\Gamma^{(s)\,NPQ}\,\sigma_1 + \frac{1}{2}\frac{1}{5!}F^{(5)}_{NPQRT}\,\Gamma^{(s)\,NPQRT}\,(i\sigma_2)\Big)\Gamma^{(s)}_M\epsilon^{(s)}\ , \qquad \text{(A.1)}
\end{aligned}
$$

where the superscript s refers to the string frame, $\sigma_i \quad i = 1,2,3$ are the Pauli matrices, H is the NSNS three-form and $F^{(1)}$, $F^{(3)}$ and $F^{(5)}$ are the RR field strengths. In (A.1) ϵ is a doublet of Majorana-Weyl spinors of positive chirality.

We can study how these equations change under a rescaling of the metric like

$$
g^{(s)}_{MN} = e^{\frac{\phi}{2}}\, g_{MN}. \qquad \text{(A.2)}
$$

In doing that it is useful to follow section 2 of [46]. Under the above change for the metric, there are some quantities which also change:

$$
\begin{aligned}
\Gamma^{(s)}_M &= e^{\frac{\phi}{4}}\,\Gamma_M\ , \\
\epsilon^{(s)} &= e^{\frac{\phi}{8}}\,\epsilon\ , \\
\lambda^{(s)} &= e^{-\frac{\phi}{8}}\,\lambda\ , \\
\psi_M &= e^{-\frac{\phi}{8}}\Big(\psi^{(s)}_M - \frac{1}{4}\Gamma^{(s)}_M\,\lambda^{(s)}\Big)\ . \qquad \text{(A.3)}
\end{aligned}
$$

The equation for the dilatino in the new frame can be easily obtained whereas in doing the same for the gravitino equation we will use that

$$
\nabla^{(s)}_M\epsilon^{(s)} = e^{\frac{\phi}{8}}\Big[\nabla_M\epsilon + \frac{1}{8}\Gamma_M^{\ \ N}(\nabla_N\phi) + \frac{1}{8}(\nabla_M\phi)\Big]\ . \qquad \text{(A.4)}
$$

After some algebra with gamma-matrices, the SUSY transformations in Einstein frame we obtain are the following ones:

$$
\begin{aligned}
\delta_\epsilon\lambda &= \frac{1}{2}\Gamma^M\big(\partial_M\phi - e^{\phi}F^{(1)}_M(i\sigma_2)\big)\epsilon + \frac{1}{4}\frac{1}{3!}\Gamma^{MNP}\big(e^{-\frac{\phi}{2}}H_{MNP}\sigma_3 - e^{\frac{\phi}{2}}F^{(3)}_{MNP}\sigma_1\big)\epsilon, \\
\delta_\epsilon\psi_M &= \nabla_M\epsilon + \frac{1}{4}e^{\phi}F^{(1)}_M(i\sigma_2)\epsilon - \frac{1}{96}\big(e^{-\frac{\phi}{2}}H_{NPQ}\sigma_3 + e^{\frac{\phi}{2}}F^{(3)}_{NPQ}\sigma_1\big)\big(\Gamma_M^{\ \ NPQ} - 9\delta^N_M\Gamma^{PQ}\big)\epsilon + \\
&+ \frac{1}{16}\frac{1}{5!}F^{(5)}_{NPQRT}\Gamma^{NPQRT}(i\sigma_2)\Gamma_M\epsilon. \qquad \text{(A.5)}
\end{aligned}
$$

In order to write the expression of the SUSY transformations, it is convenient to change the notation used for the spinor. Up to now we have considered the double spinor notation, namely the two Majorana-Weyl spinors ϵ_1 and ϵ_2 form a two-dimensional vector $\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}$. We can rewrite the double spinor in complex notation as[22] $\epsilon = \epsilon_1 - i\epsilon_2$. It is then straightforward to find the following rules to pass from complex to real spinors:

$$
\epsilon^* \leftrightarrow \sigma_3\,\epsilon\ , \qquad -i\epsilon^* \leftrightarrow \sigma_1\,\epsilon\ , \qquad i\epsilon \leftrightarrow i\sigma_2\,\epsilon\ , \qquad \text{(A.6)}
$$

[22]Notice that there is an ambiguity in the choice of the relation between complex and real spinors.

where the Pauli matrices act on the two-dimensional vector $\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}$.

B. The unflavored solutions

In this appendix we will study our BPS system of linear ordinary differential equations (2.36), (2.39) and we will find its general solution in the absence of D7-branes. Not only we will recover the solution describing a stack of D3-branes placed at the apex of the real Calabi-Yau cone over a generic Sasaki-Einstein 5-manifold M_5, preserving (at least) four supercharges, and its near-horizon limit $AdS_5 \times M_5$ dual to the (at least) $\mathcal{N} = 1$ superconformal gauge theory describing the IR dynamics on the stack of D3-branes, but also the solution describing D3-branes smeared homogeneously on a blown-up 4-cycle inside the Calabi-Yau, discussed in the paper [34] for the case of the conifold (more precisely a $\mathbb{Z}_2$ orbifold of it) and then in full generality in [35] for all Calabi-Yau cones. We will also study the unflavored limit of the general deformation of the KW model analyzed in section 3.5 and we will show that it gives rise to the two-parameter metrics found in ref. [34].

Let us look at our BPS system of linear ordinary differential equations (2.36- 2.39). We will sometimes refer to the case of the conifold for the sake of simplicity. The generalization to any Sasaki-Einstein is straightforward, the only difference being the normalization in (2.39) and in the RR 5-form field strength, related to the volume of the Sasaki-Einstein base.

First of all notice that N_f must be set to zero in the system of equations and not in our solution, since when we solved the equation for the dilaton (2.38) we supposed that $N_f \neq 0$. This allowed us to get (2.40) after shifting the radial variable.

It is easy to show that the most general solution to the BPS system when $N_f = 0$, up to redefinition of the coordinates, is the following:

$$\phi(\rho) = \phi_0 \tag{B.1}$$

$$e^{g(\rho)} = \left[e^{6\rho} + c_1\right]^{1/6} \tag{B.2}$$

$$e^{f(\rho)} = e^{3\rho}\left[e^{6\rho} + c_1\right]^{-1/3} \tag{B.3}$$

$$h(\rho) = c_2 - 4L^4 \int d\rho\, e^{-4g(\rho)} \tag{B.4}$$

$$r(\rho) = \int d\rho\, e^{f(\rho)}\ . \tag{B.5}$$

L is the common radius of AdS_5 and the Sasaki-Einstein M_5 in the solution dual to the superconformal theory, and is fixed by the number of D3 branes and the volume of the Sasaki-Einstein manifold. For $T^{1,1}$: $L^4 = \frac{27}{4}\pi N_c$.

The real integration constant c_1 discriminates different classes of solutions.

If $c_1 = 0$ then we recover the D3-branes solution (with nonzero c_2, that can be fixed to 1) or its near-horizon AdS solution (with $c_2 = 0$):

$$e^f = e^g = e^\rho = r \tag{B.6}$$

$$h = c_2 + \frac{L^4}{r^4}\ . \tag{B.7}$$

If $c_1 > 0$ the solution describes N_c smeared D3-branes on the blown-up 4-cycle of the Calabi-Yau [34, 35]. Indeed, let's consider the change of radial coordinate:

$$\left[1 + c_1 e^{-6\rho}\right]^{-1} = 1 - \frac{b^6}{r^6} \equiv k(r) \,, \tag{B.8}$$

with $r > b$. If we further identify

$$b^2 = c_1^{1/3} \,, \tag{B.9}$$

it follows that

$$e^{2g} = r^2 \tag{B.10}$$

$$e^{2f} = r^2 k(r) \tag{B.11}$$

$$e^{2f} d\rho^2 = \frac{dr^2}{k(r)} \,, \tag{B.12}$$

so that the 6-dimensional metric, which is Calabi-Yau, is

$$ds_6^2 = [k(r)]^{-1} dr^2 + \frac{k(r) r^2}{9}(d\psi + \sum_{i=1,2} \cos\theta_i d\varphi_i)^2 + \frac{r^2}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i d\varphi_i^2) \,, \tag{B.13}$$

that describes a deformation of the Calabi-Yau where a Kähler-Einstein 4-cycle is blown up at $r = b$. In order for the resolved Calabi-Yau to be smooth, an orbifolding along the U(1) fiber parameterized by ψ is usually needed. For the case of the deformation of the conifold the orbifold action is $\mathbb{Z}_2$, so that ψ ranges from 0 to 2π. The 10-dimensional metric of the solution is then:

$$ds_{10}^2 = [h(r)]^{-1/2} dx_{1,3}^2 + [h(r)]^{1/2} ds_6^2 \,, \tag{B.14}$$

with the warp factor[23]

$$h(r) = -2\frac{L^4}{b^4}\left[\frac{1}{6}\log\frac{(\tilde r^2 - 1)^3}{\tilde r^6 - 1} + \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \arctan\frac{2\tilde r^2 + 1}{\sqrt{3}}\right)\right] \,, \quad \tilde r = \frac{r}{b} \,. \tag{B.15}$$

The gauge theory dual to this local Kähler deformation of the Calabi-Yau cone is a deformation of the superconformal theory due to the insertion of a VEV of a dimension 6 operator, which is a combination of the operators $\mathrm{Tr}(\mathcal{W}_\alpha \bar{\mathcal{W}}^\alpha)^2$, $\mathcal{W}_\alpha$ being the gluino superfield [35]. The orbifold action is needed to have a dual field theory whose mesonic branch of the moduli space is (the symmetric product of N_c copies of) the resolved Calabi-Yau.

A similar analysis can be done for the solutions with $c_1 < 0$, but in that case the 6-dimensional transverse space happens to have a curvature singularity and cannot be described as an algebraic variety. Therefore the supergravity solution is not expected to describe a dual supersymmetric gauge theory.

Let us now study the unflavored limit of the general deformation of the KW solution of section 3.5. Recall that, in this case, the metric depends on three functions (f, g_1 and

[23]The additive integration constant in h is omitted in order to asymptote to $AdS_5 \times X_5$ for large values of r.

g_2) and the warp factor h. In terms of the variable ρ introduced in (3.85) the metric can be written as:

$$ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2}\, e^{2f} \Big\{ d\rho^2 + \frac{1}{6} \sum_{i=1,2} e^{2g_i - 2f} (d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2) + \\ + \frac{1}{9} \Big(d\psi + \sum_{i=1,2} \cos\theta_i\, d\varphi_i \Big)^2 \Big\} \,. \qquad \text{(B.16)}$$

The unflavored limit of the BPS system (3.84) amounts to taking $C = 0$. As in the previous case, the solution of section 3.5 is not valid in this limit (see eq. (3.87)) and one has to take $C = 0$ in the system (3.84) and integrate it again following the same steps as in section 3.5. The result can be written in terms of the function $e^{2g_2(\rho)}$, which is the solution of the following cubic equation:

$$e^{6g_2} + \frac{3}{2}\, a^2\, e^{4g_2} = e^{6\rho} + c_1 \,, \qquad \text{(B.17)}$$

where c_1 is an integration constant. In terms of $e^{2g_2(\rho)}$ the other functions of the ansatz can be written as:

$$\begin{aligned} e^{2g_2} &= e^{2g_1} + a^2 \,, \\ e^{2f} &= \frac{e^{6\rho}}{e^{4g_2} + a^2\, e^{2g_2}} \,, \\ h(\rho) &= -Q \int \frac{d\rho}{e^{4g_2} + a^2\, e^{2g_2}} + c_2 \,, \end{aligned} \qquad \text{(B.18)}$$

where Q is given by (3.83) and c_2 is a new integration constant. Let us now perform the following change of radial variable

$$e^{2g_2(\rho)} = \frac{r^2}{6} \,. \qquad \text{(B.19)}$$

Taking into account (B.17), the inverse relation between these two radial variables is:

$$e^{6\rho} = \frac{1}{216} \left[r^6 + 9a^2 r^4 - b^6 \right] \,, \qquad \text{(B.20)}$$

where we have redefined the constant c_1 as:

$$b^2 \equiv 6\, (c_1)^{\frac{1}{3}} \,. \qquad \text{(B.21)}$$

By using these relations, one can readily prove that:

$$\begin{aligned} e^{2g_1} &= \frac{1}{6}\, (r^2 + 6a^2) \,, \\ e^{2f} &= \frac{r^2}{6}\, \kappa(r) \,, \end{aligned} \qquad \text{(B.22)}$$

where the function $\kappa(r)$ is defined as follows:

$$\kappa(r) \equiv \frac{r^6 + 9a^2\, r^4 - b^6}{r^6 + 6a^2 r^4} \,. \qquad \text{(B.23)}$$

It is also easy to verify that:

$$d\rho \,=\, \frac{dr}{r\kappa(r)}\ . \tag{B.24}$$

Using these results and redefining the warp factor as $h(r)^{-\frac{1}{2}} \rightarrow h(r)^{-\frac{1}{2}}/6$, we get a metric of the form:

$$ds^2 \,=\, \big[\,h(r)\,\big]^{-\frac{1}{2}}\Big[\,dx^2_{1,3}\Big] \,+\, \big[\,h(r)\,\big]^{\frac{1}{2}}\,ds^2_6\ , \tag{B.25}$$

with ds^2_6 given by:

$$\begin{aligned} ds^2_6 \,=\,& \big[\,\kappa(r)\,\big]^{-1}dr^2 \,+\, \frac{r^2}{9}\,\kappa(r)\,(\,d\psi \,+\, \sum_{a=1,2}\cos\theta_i\,d\varphi_i\,)^2 \,+ \\ &+\frac{1}{6}\,(\,r^2 \,+\, 6a^2\,)\,(\,d\theta_1^2 \,+\, \sin^2\theta_1\,d\varphi_1^2\,) \,+\, \frac{1}{6}\,r^2\,(\,d\theta_2^2 \,+\, \sin^2\theta_2\,d\varphi_2^2\,)\ , \end{aligned} \tag{B.26}$$

while the warp factor h can be represented as:

$$h(r) \,=\, -Q\int\frac{r dr}{r^6 \,+\, 9a^2\,r^4 \,-\, b^6} \,+\, c_2\ . \tag{B.27}$$

This is the solution with two Kähler deformations found in ref. [34]: the a constant parameterizes global deformations, while the b parameter corresponds to local deformations.

C. Alternative interpretation of the IR regime

Here we put an alternative description of the IR theory as deduced from supergravity, which honestly we could not discard. It mainly arises from the analysis of the Klebanov-Witten model at small values of the string coupling, and it is based on the non-validity of the orbifold relations (2.69)-(2.72) for all values of the parameters in the KW model, that was extensively pointed out in [13]. In the whole analysis that will follow, we will consider for clarity only the case of equal gauge couplings $g_1 = g_2 \equiv g$.

The curve of conformal points in the Klebanov-Witten model is obtained by requiring the anomalous dimension of the fields A, B to be $\gamma_A(g,\tilde{\lambda}) \,=\, -1/2$, which assures $\beta_g = \beta_{\tilde{\lambda}} = 0$ ($\tilde{\lambda}$ is the dimensionless coupling from the quartic superpotential). The qualitative shape of the curve is depicted in figure 4, as well as some possible RG flows. The important feature is that there is a minimum value $g_* > 0$ that fixed points can have (due to the perturbative β_g being negative, so that $g = 0$ is an unstable IR point). One way to determine this curve of fixed points is to apply the a-maximization procedure originally spelled in [47] by using Lagrange multipliers enforcing the marginality constraints [48], and then express the Lagrange multipliers in terms of the gauge and superpotential couplings. This computation for the Klebanov-Witten model was done in [49].[24] One can show that the curve of fixed points does not pass through the origin of the space of Lagrange multipliers, which is mapped into the origin of the space of couplings (free theory). In a particular scheme the curve of fixed points is an arc of hyperbola with the major axis along $\tilde{\lambda} = 0$. The exact shape of the curve is scheme-dependent, due to scheme-dependence of

[24]We thank Sergio Benvenuti for making us aware of this method and of the literature on the subject.

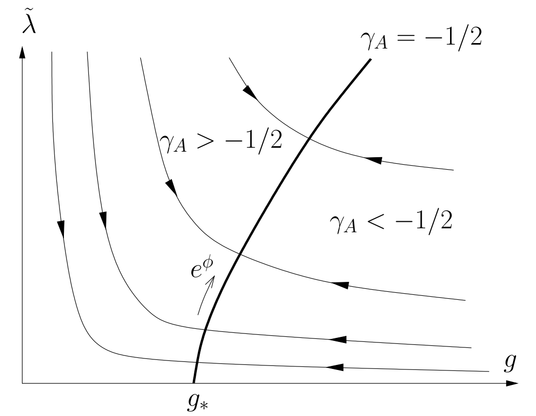


Figure 4: RG flow phase space for the Klebanov-Witten model.

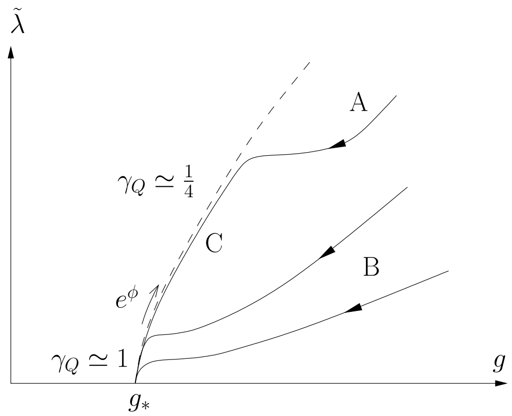


Figure 5: Klebanov-Witten model with flavors. The A-C flow has backreacting D7's in the A piece and then follows the KW line in the C piece; it corresponds to $N_f \ll N_c$. The B flow is always far from the KW line, and corresponds to $N_f \gtrsim N_c$.

the relation between Lagrange multipliers and couplings: we choose a scheme in which the Lagrange multipliers are quadratic in the couplings. This choice fixes a conic section, and it is such a hyperbola because the one-loop anomalous dimensions of the chiral superfields get a negative contribution from gauge interactions and a positive contribution from superpotential interactions. The conclusion that the curve of conformal points does not pass through the origin of the space of coupling constants is physical.

The family of KW SUGRA solutions describes the fixed curve. It is parameterized by e^ϕ that can take arbitrary values. For sufficiently large values of it, we can trust the orbifold formula:

$$\frac{g^2}{8\pi} = e^\phi \qquad \text{for} \qquad e^\phi N_c \gtrsim 1 \,. \tag{C.1}$$

The 't Hooft coupling $g^2 N_c$ is large (at least of order 1, so the theory is strongly coupled and the anomalous dimensions are of order 1) and the string frame curvature $R_S \sim 1/(e^\phi N_c)$ is small. For smaller values $e^\phi N_c \lesssim 1$, (C.1) cannot be correct: it would give small 't Hooft coupling while the gauge theory is always strongly coupled. The bottom end of the line corresponds to:

$$\{e^\phi \to 0\} \qquad \leftrightarrow \qquad \{g = g_*, \tilde{\lambda} = 0\} \,, \tag{C.2}$$

and the SUGRA curvature is large even if the field theory is still strongly coupled. Anyway some quantities, for instance the quantum dimension of A, B, are protected and do not depend on the coupling, so they can be computed in SUGRA even for small values of $e^\phi N_c$.

The supergravity solution of our system with D7-branes is in the IR quite similar to the KW geometry: the IR asymptotic background is $AdS_5 \times T^{1,1}$ (with corrections), but with running dilaton. The field theory is thus deduced to be close to KW fixed line, but running along it as $e^\phi \to 0$ in the IR. Moreover, e^ϕ controls the gravitational backreaction of the D7-branes (as well as the gauge coupling), and as soon as $e^\phi N_f \lesssim 1$ the branes behave as probes. In this regime, we expect the quantities computable from the background to be equal to the KW model ones: in particular $\gamma_A = -1/2$.

We can distinguish different regimes, starting from the UV to the IR. Depending on the values of N_c and N_f they can be either well separated or not present at all. A section of the space of couplings and some RG flows are drawn in figure 5, but one should include the third orthogonal direction h which is not plotted.

- For $1 < e^\phi$ we are in the Landau pole regime, and the dilaton (string coupling e^ϕ) is large.
- For $\frac{1}{N_f} < e^\phi < 1$ we are in a complicated piece of the flow, quite far from the KW fixed line, as in the type A-B flows of figure 5. In particular the D7-branes are backreacting. In this regime our SUGRA solution is perfectly behaved (as long as $\frac{1}{N_c} < e^\phi$).
- For $\frac{1}{N_c} < e^\phi < \frac{1}{N_f}$ (this regime exists for $N_f < N_c$) we are in a region with almost probe D7-branes,[25] so we are close to the KW line, but with large 't Hooft coupling, so we can trust (C.1). We can expect the energy/radius relation to be quite similar to the conformal one, thus we can compute the gauge β-function and deduce the flavor anomalous dimensions γ_Q. Apart from corrections, we get:

  $$\gamma_A \simeq -\frac{1}{2} \qquad R_A \simeq \frac{1}{2} \qquad\qquad \gamma_Q \simeq \frac{1}{4} \qquad R_Q \simeq \frac{3}{4}\,. \tag{C.3}$$

  The R-symmetry is classically preserved but anomalous as in supergravity. The various β-functions are computed to be

  $$\beta_g = \frac{3}{4} N_f \frac{g^3}{16\pi^2} \qquad\qquad \beta_{\tilde\lambda} \simeq 0 \qquad\qquad \beta_h \simeq 0\,. \tag{C.4}$$

  We want to stress that this regime in *not* conformal, and in fact the theory flows along the KW fixed line, as in the type C flow of figure 5. The smaller is N_f/N_c, the longer is this piece of the flow. For $N_f \gtrsim N_c$ this regime does not exist, and the theory follows type B flow of figure 5.
- For $e^\phi < \mathrm{Min}(\frac{1}{N_c}, \frac{1}{N_f})$ we are close to the end of the KW fixed line, and the gauge coupling is close to g_*. Again the D7's are almost probes. The string frame curvature is large, as in the KW model at small $g_s N_c$. Since the gauge coupling cannot go below g_*, its β-function vanishes even if the string coupling continues flowing to zero. We get in field theory:

  $$\gamma_A \simeq -\frac{1}{2} \qquad R_A \simeq \frac{1}{2} \qquad\qquad \gamma_Q \simeq 1 \qquad R_Q \simeq \frac{3}{4} \tag{C.5}$$

[25]The dual in field theory of the D7's being probes is that graphs with flavors in the loops are suppressed with respect to gauge fields in the loops, since $N_f < N_c$.

JHEP02(2007)090

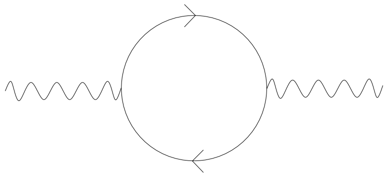


Figure 6: Flavor 1-loop correction to the gauge propagator.

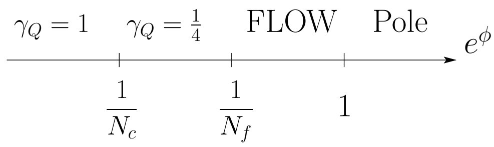


Figure 7: Regimes of KW with flavors for $N_f < N_c$.

$$\beta_g \simeq 0 \qquad\qquad \beta_{\tilde{\lambda}} \simeq 0 \qquad\qquad \beta_h = \frac{3}{4} h \, . \tag{C.6}$$

All the flows accumulate at the point $\{g = g_*, \tilde{\lambda} = 0\}$ of figure 5, but the theory is *not* conformal. In fact the coupling h always flows to smaller values, and the theory moves "orthogonal" to the figure. For this reason γ_Q and R_Q do not satisfy the relation of superconformal theories.

- The end of the flow is the superconformal point with $h = 0$ (and $g = g_*$), which should correspond to $e^{\phi} = 0$ and cannot be described by supergravity. Without the cubic superpotential one can construct a new anomaly free R-symmetry with $R_Q = 1$, by combining the previous one ($R_Q = 3/4$) with the anomalous axial symmetry which gives charge 1/4 to every flavor. This satisfies known theorems on superconformal theories. Moreover, the fact that $h \to 0$ in the far infrared realizes in field theory the incapability of resolving the D7 separation at small energies, and the flavor symmetry $S(\mathrm{U}(N_f) \times \mathrm{U}(N_f))$ is restored.

Note that when $N_f \gtrsim N_c$ and the D7-branes are probes (this is the regime $e^{\phi} < \frac{1}{N_f} < \frac{1}{N_c}$ and $g = g_*$) one could think hard to see in field theory a suppression of graphs with flavors in the loops, with respect to gauge fields in the loops. Consider the gauge propagator at 1-loop with flavors (figure 6). It is of order $g_*^2 N_f$, not suppressed with respect to the graph with gauge fields in the loop of order $g_*^2 N_c$. But if we sum all the loops with flavors, we must obtain the flavor contribution to the β-function, which for $g \simeq g_*$ and so $\gamma_Q \simeq 1$ is indeed very small.

A summary of the phase space for $N_f < N_c$ is in figure 7. The computation in [21] is valid in the region $\frac{1}{N_c} < e^{\phi} < \frac{1}{N_f}$ of the phase space.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[5] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *The supergravity dual of $N = 1$ super Yang-Mills theory*, *Nucl. Phys.* **B 569** (2000) 451 [hep-th/9909047]; *Confinement and condensates without fine tuning in supergravity duals of gauge theories*, *JHEP* **05** (1999) 026 [hep-th/9903026]; *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, *JHEP* **12** (1998) 022 [hep-th/9810126];
D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a C-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017]; *Continuous distributions of D3-branes and gauged supergravity*, *JHEP* **07** (2000) 038 [hep-th/9906194];
J. Polchinski and M. J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136;
J. Babington, D.E. Crooks and N.J. Evans, *A stable supergravity dual of non-supersymmetric glue*, *Phys. Rev.* **D 67** (2003) 066007 [hep-th/0210068]; *A non-supersymmetric deformation of the AdS/CFT correspondence*, *JHEP* **02** (2003) 024 [hep-th/0207076].

[6] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[7] I.R. Klebanov and N.A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, *Nucl. Phys.* **B 574** (2000) 263 [hep-th/9911096].

[8] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(n+m)$ *gauge theories*, *Nucl. Phys.* **B 578** (2000) 123 [hep-th/0002159].

[9] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *JHEP* **08** (2000) 052 [hep-th/0007191].

[10] S.S. Gubser, C.P. Herzog and I.R. Klebanov, *Symmetry breaking and axionic strings in the warped deformed conifold*, *JHEP* **09** (2004) 036 [hep-th/0405282].

[11] A. Dymarsky, I.R. Klebanov and N. Seiberg, *On the moduli space of the cascading* $\mathrm{SU}(m+p) \times \mathrm{SU}(P)$ *gauge theory*, *JHEP* **01** (2006) 155 [hep-th/0511254].

[12] A. Butti, M. Grana, R. Minasian, M. Petrini and A. Zaffaroni, *The baryonic branch of Klebanov-Strassler solution: a supersymmetric family of* $\mathrm{SU}(3)$ *structure backgrounds*, *JHEP* **03** (2005) 069 [hep-th/0412187].

[13] M.J. Strassler, *The duality cascade*, hep-th/0505153.

[14] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043 [hep-th/0205236].

[15] I.R. Klebanov and J.M. Maldacena, *Superconformal gauge theories and non-critical superstrings*, *Int. J. Mod. Phys.* **A 19** (2004) 5003 [hep-th/0409133].

[16] F. Bigazzi, R. Casero, A.L. Cotrone, E. Kiritsis and A. Paredes, *Non-critical holography and four-dimensional CFT's with fundamentals*, *JHEP* **10** (2005) 012 [hep-th/0505140].

[17] R. Casero, C. Núñez and A. Paredes, *Towards the string dual of $N = 1$ SQCD-like theories*, *Phys. Rev.* **D 73** (2006) 086005 [hep-th/0602027].

[18] A. Paredes, *On unquenched $N = 2$ holographic flavor*, *JHEP* **12** (2006) 032 [hep-th/0610270].

[19] S. Murthy and J. Troost, *D-branes in non-critical superstrings and duality in $N = 1$ gauge theories with flavor*, *JHEP* **10** (2006) 019 [hep-th/0606203].

[20] D. Arean, D.E. Crooks and A.V. Ramallo, *Supersymmetric probes on the conifold*, *JHEP* **11** (2004) 035 [hep-th/0408210].

[21] P. Ouyang, *Holomorphic d7-branes and flavored $N = 1$ gauge theories*, *Nucl. Phys.* **B 699** (2004) 207 [hep-th/0311084];
T.S. Levi and P. Ouyang, *Mesons and flavor on the conifold*, hep-th/0506021.

[22] T. Sakai and J. Sonnenschein, *Probing flavored mesons of confining gauge theories by supergravity*, *JHEP* **09** (2003) 047 [hep-th/0305049];
S. Kuperstein, *Meson spectroscopy from holomorphic probes on the warped deformed conifold*, *JHEP* **0503** (2005) 014 [hep-th/0411097].

[23] U. Gursoy and C. Núñez, *Dipole deformations of $N = 1$ SYM and supergravity backgrounds with* $\mathrm{U}(1) \times U(1)$ *global symmetry*, *Nucl. Phys.* **B 725** (2005) 45 [hep-th/0505100];
R.P. Andrews and N. Dorey, *Deconstruction of the Maldacena-Nunez compactification*, *Nucl. Phys.* **B 751** (2006) 304 [hep-th/0601098].

[24] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[25] G. Veneziano, *Some aspects of a unified approach to gauge, dual and Gribov theories*, *Nucl. Phys.* **B 117** (1976) 519.

[26] B.R. Greene, A.D. Shapere, C. Vafa and S.-T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nucl. Phys.* **B 337** (1990) 1.

[27] O. Aharony, A. Fayyazuddin and J.M. Maldacena, *The large-N limit of $N = 2, 1$ field theories from three-branes in F-theory*, *JHEP* **07** (1998) 013 [hep-th/9806159].

[28] M. Grana and J. Polchinski, *Gauge/gravity duals with holomorphic dilaton*, *Phys. Rev.* **D 65** (2002) 126005 [hep-th/0106014].

[29] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, *$N = 2$ gauge theories on systems of fractional D3/D7 branes*, *Nucl. Phys.* **B 621** (2002) 157 [hep-th/0107057].

[30] B.A. Burrington, J.T. Liu, L.A. Pando Zayas and D. Vaman, *Holographic duals of flavored $N = 1$ super Yang-Mills: beyond the probe approximation*, *JHEP* **02** (2005) 022 [hep-th/0406207].

[31] I. Kirsch and D. Vaman, *The D3/D7 background and flavor dependence of Regge trajectories*, *Phys. Rev.* **D 72** (2005) 026007 [hep-th/0505164].

[32] S.A. Cherkis and A. Hashimoto, *Supergravity solution of intersecting branes and AdS/CFT with flavor*, *JHEP* **11** (2002) 036 [hep-th/0210105];
B.S. Acharya, F. Denef, C. Hofman and N. Lambert, *Freund-Rubin revisited*, hep-th/0308046;
M. Gomez-Reino, S. Naculich and H. Schnitzer, *Thermodynamics of the localized D2-D6 system*, *Nucl. Phys.* **B 713** (2005) 263 [hep-th/0412015];
R. Clarkson, A.M. Ghezelbash and R.B. Mann, *New reducible membrane solutions in $D = 11$ supergravity*, *JHEP* **0404** (2004) 063 [hep-th/0404071]; *New reducible five-brane solutions in M-theory*, *JHEP* **0408** (2004) 025 [hep-th/0405148];
A. M. Ghezelbash and R. B. Mann, *Atiyah-Hitchin M-branes*, *JHEP* **0410** (2004) 012 [hep-th/0408189];

J. Erdmenger and I. Kirsch, *Mesons in gauge/gravity dual with large number of fundamental fields*, *JHEP* **0412** (2004) 025 [hep-th/0408113];
H. J. Schnitzer, *Confinement/deconfinement transition of large N gauge theories in perturbation theory with N_f fundamentals: N_f/N finite*, [hep-th/0612099].

[33] P. Candelas and X.C. de la Ossa, *Comments on conifolds*, *Nucl. Phys.* **B 342** (1990) 246.

[34] L.A. Pando Zayas and A.A. Tseytlin, *3-branes on spaces with $R \times S(2) \times S(3)$ topology*, *Phys. Rev.* **D 63** (2001) 086006 [hep-th/0101043].

[35] S. Benvenuti, M. Mahato, L.A. Pando Zayas and Y. Tachikawa, *The gauge / gravity theory of blown up four cycles*, hep-th/0512061.

[36] D.R. Morrison and M.R. Plesser, *Non-spherical horizons. I*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[37] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [hep-th/0007018].

[38] P. Olesen and F. Sannino, *$N = 1$ super-Yang-Mills from supergravity: the UV-IR connection*, hep-th/0207039.

[39] C.P. Herzog, I.R. Klebanov and P. Ouyang, *D-branes on the conifold and $N = 1$ gauge/gravity dualities*, hep-th/0205100.

[40] K. Skenderis and M. Taylor, *Holographic Coulomb branch vevs*, *JHEP* **08** (2006) 001 [hep-th/0604169].

[41] M. Bertolini and P. Merlatti, *A note on the dual of $N = 1$ super-Yang-Mills theory*, *Phys. Lett.* **B 556** (2003) 80 [hep-th/0211142].

[42] M. Grana, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [hep-th/0509003].

[43] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074 [hep-th/0604007];
J. Gomis and C. Romelsberger, *Bubbling defect CFT's*, *JHEP* **0608** (2006) 050 [hep-th/0604155];
O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **0606** (2006) 026 [hep-th/0604133];
S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **0605** (2006) 037 [hep-th/0603208];
O. Lunin and J. M. Maldacena, *Deforming field theories with $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry and their gravity duals*, *JHEP* **0505** (2005) 033 [hep-th/0502086].

[44] J.H. Schwarz, *Covariant field equations of chiral $N = 2, D = 10$ supergravity*, *Nucl. Phys.* **B 226** (1983) 269.

[45] L. Martucci, J. Rosseel, D. Van den Bleeken and A. Van Proeyen, *Dirac actions for D-branes on backgrounds with fluxes*, *Class. and Quant. Grav.* **22** (2005) 2745 [hep-th/0504041].

[46] A. Strominger, *Superstrings with torsion*, *Nucl. Phys.* **B 274** (1986) 253.

[47] K. Intriligator and B. Wecht, *The exact superconformal R-symmetry maximizes a*, *Nucl. Phys.* **B 667** (2003) 183 [hep-th/0304128].

[48] D. Kutasov, *New results on the 'a-theorem' in four dimensional supersymmetric field theory*, hep-th/0312098.

[49] S. Benvenuti and A. Hanany, *Conformal manifolds for the conifold and other toric field theories*, *JHEP* **08** (2005) 024 [hep-th/0502043];
S. Benvenuti and M. Kruczenski, *Semiclassical strings in Sasaki-Einstein manifolds and long operators in $N = 1$ gauge theories JHEP* **0610** (2006) 051 [hep-th/0505046].